

Dynamic Moral Hazard and Stopping*

Robin Mason[†]

Juuso Välimäki[‡]

3 January 2011

Abstract

We analyse a simple model of dynamic moral hazard in which there is a clear and tractable trade-off between static and dynamic incentives. In our model, a principal wants an agent to complete a project. The agent undertakes unobservable effort, which affects in each period the probability that the project is completed. We characterise the contracts that the principal sets, with and without commitment. We show that with full commitment, the contract involves the agent's value and wage declining over time, in order to give the agent incentives to exert effort. The long-run levels of the value and wage depend on the relative discount rates of the principal and agent. We also characterise the set of sequentially rational equilibria, where the principal has no commitment power.

Keywords: Principal-agent model, continuous time, moral hazard, project completion.

JEL classification: C73; D82; J31.

*We are grateful to Matti Liski, as well as numerous seminar participants, for many helpful comments.

[†]University of Exeter and CEPR. University of Exeter Business School, Streatham Court, Rennes Drive, Exeter EX4 4PU, UK, mason.robin@gmail.com. Robin Mason acknowledges financial support from the ESRC under Research Grant RES-062-23-0925.

[‡]Aalto University School of Economics and HECER. Arkadiankatu 7, FI-00100 Helsinki, juuso.valimaki@hse.fi.

1 Introduction

It has been estimated that approximately 90% of American companies use, to some extent at least, agency firms to find workers; see [Fernandez-Mateo \(2003\)](#). According to [Finlay and Coverdill \(2000\)](#), between 13% and 20% of firms use private employment agencies “frequently” to find a wide variety of workers. The proportion is higher when searching for senior executives. Recent surveys in the pharmaceutical sector estimate almost two-thirds of senior executive hires involve a ‘headhunter’. Typically, it takes a number of months to find and recruit a candidate; for example, the same pharmaceutical source states that the average length of time taken to fill a post is between four and six months from the start of the search. See [Pharmafocus \(2007\)](#). While payment happens in a variety of ways, most contracts involve an element of payment when a candidate is placed successfully. See [Finlay and Coverdill \(2000\)](#). There is risk attached to search: between 15% and 20% of searches in the pharmaceutical sector fail to fill a post ([Pharmafocus \(2007\)](#)).

Residential real estate accounts for a large share of wealth—on average, around 30% over the period 1983–2007 in the U.S.,¹ and 39% in the UK in 1999, according to the Office of National Statistics.² The value of residential house sales in England and Wales between July 2006 and July 2007 was of the order of 10% of the UK’s GDP.³ According to the [Office of Fair Trading \(2004\)](#), over nine out of ten people buying and selling a home in England and Wales use a real estate agent. The median time in the UK to find a buyer who eventually buys the property is 137 days. (The details of the UK market are given in [Merlo and Ortalo-Magné \(2004\)](#) and [Merlo, Ortalo-Magné, and Rust \(2006\)](#).) The real estate agent can affect buyer arrival rates through exerting marketing effort. But there is also exogenous risk (such as general market conditions) that affect the time to sale. In most cases, the real estate agent is paid only on completion of the sale.

In both of these examples, a principal hires an agent to complete a project. The prin-

¹Source: own calculations from the 1983, 1989, 1992, 1995, 1998, 2001, 2004 and 2007 U.S. Survey of Consumer Finances.

²Wealth and Assets Survey 2006/08.

³More recent figures are likely to be less impressive.

principal gains no benefit until the project is completed. The agent can affect the probability of project completion by exerting effort. The principal's objective is to provide the agent with dynamic incentives to exert effort. We assume that the agent is risk neutral and subject to a limited liability constraint (in other words, the agent cannot make payments to the principal). By assuming away risk neutrality, we can argue that our dynamic results arise from the optimal management of the agent's intertemporal surplus.

The task of this paper is to analyse the dynamic incentives that arise in these settings and the contracts that are written as a result. Dynamics matter for both sides. For the agent, its myopic incentives are to equate the marginal cost of effort with the marginal return. But if it fails to complete the project today, it has a further chance tomorrow. This continuation value means that the forward-looking agent reduces its current effort, substituting towards future effort. Hence this dynamic factor, all other things equal, tends to reduce the agent's effort towards project completion. Similarly, the principal's myopic incentives trade off the marginal costs of inducing greater agent effort (through higher payments) with the marginal benefits. But the principal also knows that the project can be completed tomorrow; all other things equal, this tends to lower the payment that the principal pays today for project completion. On the other hand, the principal also realises that the agent faces dynamic incentives; this factor, on its own, tends to increase the payment to the agent.

Our modelling approach allows us to resolve these different incentives to arrive at analytical conclusions. We start by analysing the stationary game where the principal sets a constant wage for the agent once and for all. If the agent expects the wage rate to remain constant for all future periods, then it would be in the best interest of the principal to offer a temporary wage increase. This simple observation shows that it is in the principal's best interest to offer non-stationary wage profiles over time. There are two separate scenarios to compare: in the first, the principal can commit to a full sequence of future wages; in the second, she has to make offers that are sequentially rational (given the expectations of the agent).

In the full commitment case, we distinguish between two cases. When the agent is

more patient than the principal, the principal can induce the agent to take an arbitrarily high effort with an arbitrarily low cost. Because of the differences in the discount factors, a wage payment with a fixed present value to the agent is less costly to the principal (in present value terms) when the payment is delayed. Hence the principal can promise a high reward for success at negligible cost in this case through delayed payments.

The moral hazard problem has bite, then, only when the agent is no more patient than the principal. We show that in this case, the agent's wage and effort in the full commitment solution are falling over time. This is the only way in which the principal can resolve in its favour the trade-off between static incentives (which call for a high current wage) and dynamic incentives (which call for lower future wages). When the principal is equally patient as the agent, the agent's wage and effort, must converge to zero. On the other hand, when the principal is more patient than the agent, the agent's wage and effort converge to strictly positive levels. In this case, the principal can rely on the agent's impatience to provide incentives for current effort. It is clear that solutions where the effort level converges to zero cannot be supported as sequentially rational solutions of the game and, as a result, we see that the principal gains from her ability to commit.⁴

We also characterise sequentially rational equilibria, in which the principal has no commitment ability but instead sets payments period-by-period. We show that the stationary sequentially rational equilibrium, in which the principal offers the same wage in each period and the agent takes the same action, yields the lowest equilibrium payoff to the principal. We characterise also the equilibrium yielding the highest payoff to the principal, linking it in an intuitive way to the full commitment solution.

We conclude the analysis by considering how the main results might change when project quality matters. (For most of the paper, we assume that the completed project yields a fixed and verifiable benefit to the principal.)

At first glance, our results look similar to those in papers that look at unemployment

⁴In the case where the agent is more patient than the principal, any contract could be improved by delaying payments for completed projects. This delay would exploit the difference in the intertemporal rates of substitution between the agent and the principal. Our assumption of limited liability on the part of the agent makes the opposite intertemporal trades impossible.

insurance: see e.g., [Shavell and Weiss \(1979\)](#) and [Hopenhayn and Nicolini \(1997\)](#). In these papers, a government must make payments to an unemployed worker to provide a minimum level of expected discounted utility to the worker. The worker can exert effort to find a job; the government wants to minimise the total cost of providing unemployment insurance. [Shavell and Weiss \(1979\)](#) show that the optimal benefit payments to the unemployed worker should decrease over time. [Hopenhayn and Nicolini \(1997\)](#) establish that the government can improve things by imposing a tax on the individual when it finds work.

Some aspects of our analysis are similar: for example, that the principal's optimal payment under full commitment decreases over time (c.f., decreasing unemployment benefits over time). The economic forces at work are different, of course. In the unemployment insurance papers, the need to smooth over time the consumption of the risk-averse worker constrains the incentives that can be offered through unemployment benefits. In this paper, the risk neutral agent smooths its effort over time; the principal sets a declining wage to counteract this incentive. But other aspects of our analysis are quite different. First, we go beyond the unemployment insurance papers by analysing the full commitment solution for all possible values of the discount rates of the principal and agent. This shows the importance of relative levels of patience in determining intertemporal incentives. Secondly, we provide a full characterisation of the set of sequentially rational equilibria, showing the benefits of commitment for the principal.

Our work is, of course, related to the broader literature on dynamic moral hazard problems: particularly the more recent work on continuous-time models. This literature has demonstrated in considerable generality the benefits to the principal of being able to condition contracts on the intertemporal performance of the agent. By doing so, the principal can relax the agent's incentive compatibility constraints. See e.g., [Malcomson and Spinnewyn \(1988\)](#) and [Laffont and Martimort \(2002\)](#). More recently, [Sannikov \(2007\)](#), [Sannikov \(2008\)](#) and [Williams \(2006\)](#) have analysed principal-agent problems in continuous time. For example, in [Sannikov \(2008\)](#), an agent controls the drift of a diffusion process, the realisation of which in each period affects the principal's pay-

off. When the agent's action is unobserved, Sannikov characterises the optimal contract quite generally, in terms of the drift and volatility of the agent's continuation value in the contract.

An immediate difference between this paper and e.g., [Sannikov \(2008\)](#) is that we concentrate on project completion. We think this case is of independent interest for a number of different economic applications. But we also think that our setting, while less general in some respect than Sannikov's, serves to make very clear the intertemporal incentives at work. In Sannikov's models, incentives may be back or front loaded: the agent's value can drift either upward or downward. In our model, the agent's value drifts in only one direction: downwards. This is a direct consequence of the project completion setting. This same setting allows us to analyse the full time path of wages and actions, which also drift downwards. We can deal with different discount rates for the principal and the agent, leading to a distinction in terms of the asymptotic behaviour of the agent's value, wage and effort. In our view, this set-up allows a particularly clear demonstration of the long-run dynamics of the optimal contract.

Our work is also related to [Bergemann and Hege \(2005\)](#). In that paper, an entrepreneur seeks funding from an investor to carry out a risky project. The quality of the project (the likelihood of success) is unknown: both parties learn about this over time. The probability of success is affected also by the entrepreneur's effort, which may be unobserved by the investor. Bergemann and Hege analyse the equilibrium dynamics of the contract signed between the entrepreneur and investor. In certain cases, they find dynamics that are similar to those that we derive. There are two key differences between the papers, however. First, we assume that the principal makes the contract offer; in contrast, Bergemann and Hege assume that the agent makes the offer. Secondly, we allow for the principal and the agent to have different discount rates. As we shall show, this makes an important difference to the equilibrium contract.

The rest of the paper is structured as follows. [Section 2](#) lays out the basic model. [Section 3](#) looks at the situation when the principal commits to a wage that is constant over time. This analysis gives a clear intuition for the properties of the wage that the

principal sets when it has full commitment power (and so can commit to a non-constant wage). The latter is analysed in section 4. Section 5 examines sequentially rational equilibria. Section 6 considers the issues that arise when the agent can affect the quality of the completed project. Our overall conclusions are stated in section 7. An appendix contains the proofs.

2 The Model

We consider the continuous-time limit of a model where an agent must exert effort in any period in order to have a positive probability of success in a project. We assume that the effort choices of the agent are unobservable but the success of the project is verifiable; hence payments can be contingent only on the event of success or no success.

The principal and the agent are risk neutral. The agent is credit constrained so that payments from principal to agent must be non-negative in all periods. (Otherwise the solution to the contracting problem would be trivial: sell the project to the agent.) In fact, the agent could be allowed to be risk averse. The key assumption for our analysis is that the agent's value from contracting is positive, which here is a result of limited liability.

The probability of success when the agent exerts effort a within a time interval of length Δ is $a\Delta$ and the cost of such effort is $c(a)\Delta$. We make the following assumption about the cost function.

Assumption 1 1. $c'(a) > 0$, $c''(a) > 0$ and $c'''(a) \geq 0$ for all $a \geq 0$.

2. $ac'(a) - c(a) - a^2c''(a) \leq 0$ for all $a \geq 0$.

3. $c(0) = 0$ and $\lim_{a \rightarrow \infty} c'(a) = \infty$.

This assumption is satisfied e.g., for quadratic costs: $c(a) = \gamma a^2$, where $\gamma > 0$.

Success is worth $v \geq 0$ to the principal. The principal and the agent have respective discount rates of r_P and r_A .

3 Commitment to constant wage offers

In this section, we look at how a principal who has commitment power will set a constant wage. The main purpose of this analysis is to provide a benchmark to contrast the full commitment solution (section 4) and sequentially rational equilibria (section 5).

The simplest contract between the principal and the agent takes the form of a contingent on a success. We consider contracts in which the principal pays a constant wage $w \geq 0$ whenever a success takes place, and nothing if there is no success. (The latter is clearly optimal: for incentive provision, only the extra payment that results from a success matters; so it would make no sense to make any payment to the agent before project completion.) Let $W(w)$ denote the expected payoff to the agent when the principal offers the constant wage w . If the agent chooses effort level a for a time interval Δ at cost $c(a) \Delta$, she completes the project with probability $a\Delta$ and receives the wage payment w . If the project is not completed no wage is paid, and both parties face the same problem in period $t + \Delta$. Therefore, we have

$$W(w) = \max_a \{a\Delta w - c(a) \Delta + e^{-r_A \Delta} (1 - a\Delta) W(w)\}.$$

In the limit as $\Delta \rightarrow 0$, this becomes

$$r_A W(w) = \max_a \{a(w - W(w)) - c(a)\}. \quad (1)$$

The first-order condition to this problem (which is necessary and sufficient by our assumptions on the cost function) is

$$c'(a) = w - W(w). \quad (2)$$

This condition requires that the marginal cost of effort in the current period is equal to the marginal gain from a success today, net of the opportunity to try again tomorrow. For a given current wage, a higher continuation value makes the agent less willing to work in the current period.

In the model with a constant wage, an increase in the current wage implies an increase in all future wages, and therefore a higher continuation payoff to the agent. Totally differentiating equation (2) with respect to a and w gives

$$a'(w) = \frac{1 - W'(w)}{c''(a)} = \frac{r_A}{(r_A + a) c''(a)}. \quad (3)$$

where we have used envelope theorem in (1) to get the last equality.

Using similar reasoning, we can express the seller's value as follows:

$$V^S = \max_w \left\{ \frac{a(w)(v-w)}{r_P + a(w)} \right\},$$

in the limit as $\Delta \rightarrow 0$. The first-order condition for the principal is then:

$$w = v - \frac{a(w)(r_P + a(w))}{r_P a'(w)} = v - \frac{a(w)(r_P + a(w))(r_A + a(w))}{r_P r_A} c''(a(w)), \quad (4)$$

where we have used equation (3) for the last equality. Under our assumption that $c''(a) \geq 0$ and $c'''(a) \geq 0$, the right-hand side in equation (4) is decreasing in w while the left-hand side is increasing. Therefore the solution w^S to equation (4) is unique. Since the problem has an interior solution, we know that the solution to the first-order condition also solves the principal's problem. Let $a^S \equiv a(w^S)$ and $W^S \equiv W(w^S)$.

It is easy to see that even though the environment is completely stationary, the principal benefits from setting non-stationary wages. The key difference is that a change in current wages does not imply an increase in future wages. As a result, a temporary wage increase induces a larger increase in the agent's optimal effort. We formalize this simple observation in the proposition.

Proposition 1 *A temporary wage increase at t improves the principal's payoff relative to the optimal stationary wage schedule.*

Proof. See the appendix. ■

This proposition gives a first indication of the advantages of offering more flexible

wage schedules. Of course it does not give us very good indication of what the qualitative properties of the optimal wages might be. We turn to the problem of optimal wage determination in the following section. Proposition 1 also points out the importance of commitment. If a current bonus makes the worker expect a bonus in future periods, then it is clear that flexible wages are less attractive to the principal. We determine the set of wage contracts that can be supported as sequentially rational equilibria in a wage setting game without commitment possibilities in section 5.

4 Wages with full commitment

We have established that a temporary wage increase from the constant commitment level would be desirable for the principal. If the principal can commit at the *ex ante* stage to an arbitrary wage schedule, a one-off bonus followed by a constant wage is, of course, a feasible wage path. There is no reason to believe, however, that such a simple path would be optimal amongst all possible paths. In this section, we determine the optimal wage contract when the principal has full commitment.

As argued in the introduction, our model is of interest for the case where the agent is less patient than the principal. Hence we assume from now on that

$$r_A \geq r_P.$$

We consider contracts of the form where the principal pays $w(t) \geq 0$ to the agent if a success takes place in time period t and nothing if there is no success. As we have argued previously, this is the only form of contract that the principal will use.

We can analyse the problem in one of two ways. The first is to specify an optimal control problem to maximize the principal's initial value, but with the continuation values of the principal and the agent as state variables. The second is to use the method developed by Spear and Srivastava (1987) and Phelan and Townsend (1991) and write the optimal contract in terms of the agent's continuation value as the state variable.⁵

⁵In earlier versions of this paper, we used this second approach.

Both will give the same answer, but it turns out to be easier to use the optimal control approach.

The first step, then, is to determine the evolution of the two state variables in the problem, $W(t)$ and $V(t)$. Consider an arbitrary reward function $w(t)$. The agent's Bellman equation is given by:

$$r_A W(t) = \max_a \{a(w(t) - W(t)) - c(a) + \dot{W}(t)\}.$$

The agent's first-order condition is:

$$c'(a(t)) = w(t) - W(t). \tag{5}$$

The convexity of $c(\cdot)$ ensures that this is necessary and sufficient for an optimum. Substituting into the Bellman equation gives

$$\dot{W}(t) = r_A W(t) - (a(t)c'(a(t)) - c(a(t))). \tag{6}$$

Equation (6) describes how the agent's value must evolve in equilibrium. Since $c(\cdot)$ is convex, W cannot increase in percentage terms faster than the discount rate r_A . (In fact, we shall see that in equilibrium, W decreases.)

The principal's Bellman equation is

$$r_P V(t) = \max_a \left\{ a(t)(v - V(t) - W(t) - c'(a(t))) + \dot{V}(t) \right\}$$

where equation (5) has been used to substitute for the wage. When re-arranged, this becomes

$$\dot{V}(t) = (r_P + a(t))V(t) - a(t)(v - W(t) - c'(a(t))). \tag{7}$$

The optimal control formulation of the full commitment problem is

$$\begin{aligned} & \max_{\{a(t), W(0)\}} V(0) \quad \text{subject to} \\ & \dot{V}(t) = (r_P + a(t))V(t) - a(t)(v - W(t) - c'(a(t))), \\ & \dot{W}(t) = r_A W(t) - (a(t)c'(a(t)) - c(a(t))). \end{aligned} \tag{8}$$

There are implicit constraints on the state variables V and W , which both must be non-negative. There is a similar non-negativity constraint on the principal's control variable, $a(t)$, the agent's effort level. We will not incorporate these constraints explicitly in the problem, but will ensure that they are satisfied when characterizing the optimal solution.

By Pontryagin's principle, there must exist continuous functions (the co-state multipliers) attached to the first and the second constraint respectively, denoted by $\lambda(t)$ and $\mu(t)$, so that when the Hamiltonian

$$\begin{aligned} \mathcal{H}(t) \equiv & \lambda(t)((r_P + a(t))V(t) - a(t)(v - W(t) - c'(a(t)))) \\ & + \mu(t)(r_A W(t) - (a(t)c'(a(t)) - c(a(t)))) \end{aligned}$$

is defined, the necessary conditions for optimality are given by:

$$\mu(0) = 0; \tag{9}$$

$$\frac{\partial \mathcal{H}}{\partial a} = -\lambda(v - W - V - c'(a) - ac''(a)) - \mu ac''(a) = 0; \tag{10}$$

$$\dot{\lambda} = (r_P + a)\lambda - \frac{\partial \mathcal{H}}{\partial V} = 0; \tag{11}$$

$$\dot{\mu} = (r_P + a)\mu - \frac{\partial \mathcal{H}}{\partial W} = (r_P - r_A + a)\mu - a\lambda. \tag{12}$$

In these conditions, we have suppressed the time indices for notational convenience. We have also expressed the problem in current values terms, as the last two conditions (11) and (12) make clear. Also, in those conditions, the principal's effective discount rate is $r_P + a$, since in any interval $[t, t + \Delta t]$, there is a probability $a(t)\Delta t$ that the agent completes the project and the problem stops.

There is also a transversality condition; the form of this condition depends on whether it is optimal for the principal to stop the problem (i.e., fire the agent) in finite time; or whether an infinite horizon is optimal. Our first result in this section considers this question, showing that it is not optimal to choose a finite stopping time.

Proposition 2 *A finite stopping time is not optimal in the full commitment problem when $r_P \leq r_A$.*

Proof. See the appendix ■

Proposition 2 shows that sufficient incentives can be provided to the agent by reducing his wage over time, without resorting to firing him. This is in contrast to e.g., the model of Sannikov (2008), where the possibility of firing (and retiring) are important parts of the incentive scheme. The result will allow us to establish unambiguously the dynamics of the optimal contract. We shall see that the dynamics are such that once the agent's value starts to increase, it must subsequently always increase. (That is, if $\dot{W}(t) > 0$ at some t , then $\dot{W}(t') > 0$ for all $t' > t$). But this is inconsistent with the transversality condition when the horizon is infinite, which essentially requires the optimal path to be bounded. This implies that the agent's value must be decreasing along the optimal path.

We now establish this result in detail. The appropriate transversality condition in this case can be stated formally as follows. If $V^*(t)$ and $W^*(t)$ are optimal trajectories of the state variables, then

$$\lim_{t \rightarrow \infty} \lambda(t)(V^*(t) - V(t)) \leq 0, \quad \lim_{t \rightarrow \infty} \mu(t)(W^*(t) - W(t)) \leq 0$$

for all feasible trajectories $V(t)$ and $W(t)$. (See Kamihigashi (2001).) Less formally, the system must converge to a steady state in which the controls and states are stationary and bounded.

We already have expressions for the dynamics of V and W , as well as the co-state variables. We now derive the differential equation for the optimal control a . To simplify the derivation, we set λ (which is a constant, from equation (11)) equal to -1 ; this is

without loss of generality. Then, differentiate the first-order condition (10) with respect to time:

$$\begin{aligned} (c''(a(t)) + (\mu(t) + 1)(c''(a(t)) + a(t)c'''(a(t))))\dot{a}(t) \\ = -\dot{V}(t) - \dot{W}(t) - a(t)c''(a(t))\dot{\mu}(t). \end{aligned} \quad (13)$$

We establish in the proof of proposition 3 (see the appendix) that the Hamiltonian \mathcal{H} is equal to zero along the optimal path. This means that we can write \dot{V} as $\mu\dot{W}$. Doing this, and using equations (6) and (12) to substitute in for $\dot{W}(t)$ and $\dot{\mu}(t)$, gives the following differential equation for the dynamics of $a(t)$:

$$\begin{aligned} (c''(a(t)) + (\mu(t) + 1)(c''(a(t)) + a(t)c'''(a(t))))\dot{a}(t) \\ = -(\mu(t) + 1)(r_A W - (a(t)c'(a(t)) - c(a(t))) + a^2 c''(a)) \\ + a^2(t)c''(a(t))(r_P - r_A)\mu(t). \end{aligned} \quad (14)$$

We are now able to characterise the dynamics of the optimal commitment contract when $r_A \geq r_P$.

Proposition 3 *In the full commitment solution when $r_A \geq r_P$, the agent's continuation value $W(t)$, the optimal wage profile $w(t)$, and the agent's effort level $a(t)$ are all decreasing over time. If $r_P = r_A$, then the continuation value, wage and effort levels converge to zero. If $r_P < r_A$, then the wage and effort levels converge to strictly positive levels. The initial effort level, and hence all levels, are below the myopic effort a^M (defined by $v - c'(a^M) - a^M c''(a^M) = 0$).*

Proof. See the appendix. ■

Figures 1 and 2 illustrate the proof of the proposition. In the shaded area in the figures, \dot{a} and \dot{W} are non-positive. The optimal initial point must lie on the portion of the $W_{V'=0}(a)$ curve that is in bold. Any optimal path from that portion of the curve must move into the shaded area; hence \dot{a} and \dot{W} must be non-positive along the entire path.

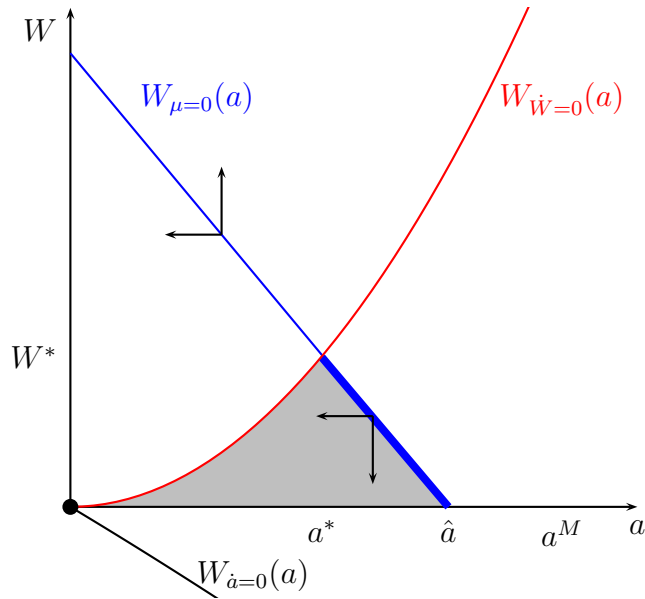


Figure 1: Phase diagram for the problem with full commitment with $r_P = r_A$

Possible steady states are marked with a dot; clearly, they must lie on the curve along which $\dot{W} = 0$. If $r_P = r_A$, then only one steady state exists: the origin, with $a = W = 0$. Otherwise, two steady states exist (as shown in the figure). The optimal path converges to the steady state with $a = W = 0$, if $r_P \geq r_A$. Otherwise, it converges to a steady state with strictly positive levels of a and W .

These results are intuitive. Because the agent has increasing marginal cost of effort, he looks to smooth his effort over time: to substitute away from current effort toward future effort. Limited liability means that the agent earns positive rents from the contract. The principal uses the dynamics of these rents to provide the agent with incentives to exert effort. In particular, the full commitment contract ensures that the agent's continuation value is decreasing in equilibrium. By these means, the principal gives the agent incentives to exert current effort. The continuation value is driven downwards by a decreasing wage; the agent's effort also decreases over time. When the agent is as patient as the principal, the principal has to drive the agent's value, wage and hence effort down to zero in the long-run in order to generate incentives for effort. But when the agent is less patient, the principal can rely on the agent's impatience to generate incentives. In this case, the long-run value, wage and effort are all strictly positive. In all cases, the principal induces

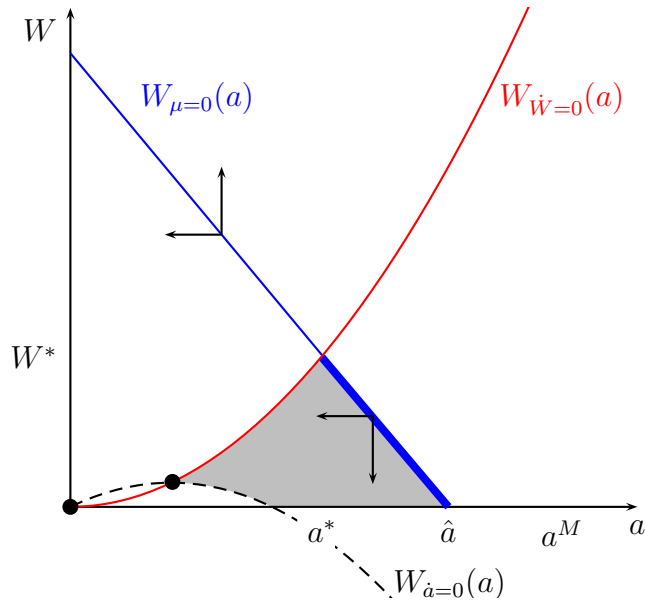


Figure 2: Phase diagram for the problem with full commitment with $r_P < r_A$

less effort from forward-looking agent: equilibrium effort is less than the myopic level.

The phase diagrams can also be used to assess the comparative dynamics of the contract. For example, consider the case when $r_A > r_P$, in figure 2; and look at the limit as $r_A \rightarrow \infty$, so that the agent becomes myopic. The curve $W_{\mu=0}(a)$ is unaffected by changes in r_A . The curve $W_{\dot{W}=0}(a)$, however, shifts downward as r_A increases, in the limit running along the horizontal axis. It is straightforward to show that the curve $W_{\dot{a}=0}(a)$ shifts so that in the limit, it passes through the point on the horizontal axis where the two other curves intersect. Hence, with a myopic agent, the optimal contract has no dynamics: the agent's effort remains constant at its initial level, and his value is 0. This is perfectly intuitive. Note also that the effort in this case is still below the myopic level a^M . This is because the principal takes into account her continuation value when writing the contract.

The final result in this section examines whether the agent completes the project in finite time.

Proposition 4 *When the principal faces an impatient agent ($r_P < r_A$), the project is completed in finite time almost surely. When the principal faces an equally patient agent ($r_P = r_A$), if the cost function is not too convex (i.e., if $a^2/c''(a) = O(a^\beta)$ for some*

$\beta < 2$), then the probability that the project is completed in finite time is less than one. Otherwise, if $\beta \geq 2$, then the project is completed in finite time almost surely.

Proof. See the appendix. ■

The proposition follows immediately when the agent is relatively impatient, since his effort is always strictly positive. The proof is more subtle when the agent is relatively patient, since in this case, his effort declines to zero. The proof relies on looking at the rate of decline of the agent's effort as the steady state is approached. The intuition is that if costs are highly convex, it is costly for the principal to induce the agent to have big differences in her effort across time. As a consequence, the agent's effort decreases relatively slowly in equilibrium (at least near the steady state). This means that $\int_0^\infty a(t)dt$ is unbounded and so the project completes almost surely. But when costs are not too convex, it is not too costly for the principal to induce high current effort and lower future effort. This means that the agent's effort decreases relatively quickly in equilibrium (near the steady state); consequently, $\int_0^\infty a(t)dt$ is bounded and the probability that the project completes is less than 1.

5 Wages without commitment

In the previous section, we assumed that the principal can commit to an entire path of future wages. While full commitment is the standard assumption in static mechanism design problems, relaxing this assumption is quite natural in dynamic settings. When the agent is as patient as the principal, the optimal commitment wage decreases towards zero. This implies that the optimal effort exerted by the agent and the continuation value to the principal also converge to zero. In these circumstances, the principal would be tempted to offer a temporary bonus for the worker from completing the task. In order to analyze the principal-agent problem when such bonuses are possible, one must make use of the full machinery of repeated extensive-form games to pin down the principal's and the agent's behaviour after all possible wage offers.

In this section, we make the opposite extreme assumption on commitment. The

principal can promise only temporary spot wages in each period that condition on whether the project was successfully completed. In order to lay out the structure of these contracts, we start with a discrete-time formulation, letting Δ denote the period length in the equilibrium computations; and then pass to the continuous-time limit. The details of the strategic structure of this dynamic contracting game are laid out in the appendix.

A strategy for the principal is a pair of history contingent wage offers $(\mathbf{w}^s(h^t), \mathbf{w}^f(h^t))$, where the first term gives the promised wage in period t after publicly observed history h^t (of past wage offers) if the project is completed, and the second term gives the corresponding wage when there is no success. A strategy for the agent is a history-dependent action choice $\mathbf{a}(h^t)$ that assigns an effort level to each history of wage offers (including the current period ones). We use the pair (\mathbf{w}, \mathbf{a}) without any arguments to denote the functions assigning actions to histories. As usual, we also let $\mathbf{w}|_{h^t}$ and $\mathbf{a}|_{h^t}$ denote the continuation strategies of the two players following an arbitrary history h^t .

A sequentially rational equilibrium in our game is a pair (\mathbf{w}, \mathbf{a}) such that $\mathbf{w}|_{h^t}$ is optimal given $\mathbf{a}|_{h^t}$ for all h^t , and $\mathbf{a}|_{h^t}$ is optimal given $\mathbf{w}|_{h^t}$ for all h^t . From this point onwards, we consider only sequentially rational equilibria. Since the underlying contracting game is stationary in the sense that the continuation game after each history (conditional on no success) is strategically equivalent to the original game, we may view the set of equilibrium continuation wage setting strategies in a recursive manner. The set of sequentially rational equilibrium strategies does not depend on h^t . We denote the set of equilibrium strategies by \mathcal{E} . Then every equilibrium continuation strategy $(\mathbf{w}, \mathbf{a}) \in \mathcal{E}$ after history h^t can be written as⁶

$$(\mathbf{w}, \mathbf{a}) = ((\mathbf{w}^s(h^t), \mathbf{w}^f(h^t), \mathbf{a}(h^t, w^s, w^f)), (\widehat{\mathbf{w}}, \widehat{\mathbf{a}})) \text{ for some } (\widehat{\mathbf{w}}, \widehat{\mathbf{a}}) \in \mathcal{E}.$$

In this section, our task is to characterize pairs $(V(\mathbf{w}, \mathbf{a}), W(\mathbf{w}, \mathbf{a}))$ of payoffs induced by some $(\mathbf{w}, \mathbf{a}) \in \mathcal{E}$. The induced payoffs are computed in a straightforward manner as detailed in the appendix.

⁶Since our stage game is in extensive form, sequential rationality also implies that for any w_t , the continuation $(\widehat{\mathbf{w}}, \widehat{\mathbf{a}})$ constitutes an equilibrium in the game.

We start by making some preliminary observations on the equilibrium payoffs for the principal. First we show that the game has a unique stationary equilibrium. By a stationary equilibrium, we simply mean a wage-setting strategy for the principal where $\mathbf{w}^f(h^t)$, $\mathbf{w}^s(h^t)$ and $\mathbf{a}(h^t)$ are independent of h^t . In other words, the principal offers the same wage contracts in all periods and the agent conditions her current effort choice on the current wage offers and the stationary wage strategy for the future periods.

Proposition 5 *The game has a unique stationary equilibrium where $\mathbf{w}^s(h^t) = w^{SR} > 0$, and $\mathbf{w}^f(h^t) = 0$ for all h^t and histories and $\mathbf{a}(h^t) = a^{SR} > 0$ after all histories h^t where $\mathbf{w}^s(h^t) = w^{SR} > 0$, and $\mathbf{w}^f(h^t) = 0$.*

Proof. See the appendix. ■

Let the payoffs in this equilibrium be V^{SR} and W^{SR} .

Our second result shows that the principal's minimal payoff in the set of sequentially rational equilibria is induced by the stationary equilibrium. In the course of proving this result, we show that all wage paths that generate higher payoffs to the principal than the stationary equilibrium can be supported as sequentially rational equilibria. This is turns out to be quite useful for characterizing the best sequentially rational equilibrium for the principal. In particular, we can use the laws of motion established in the previous Section to describe the sequentially rational equilibrium path with the highest payoff to the principal.

Proposition 6 *All sequentially rational equilibria induce a payoff of at least V^{SR} to the principal.*

Proof. See the appendix. ■

It should be noted that even though the worst equilibrium payoff to the principal can be supported as a stationary equilibrium, the same is not true of the best payoff. In the proof of Proposition 6, a key step establishes that a relatively high equilibrium continuation payoff to the principal can be reduced by an essentially unconditional monetary transfer from the principal to the agent. To get a similar mechanism to work for the best

equilibria from the principal's point of view, we would need a mechanism that transfers continuation value unconditionally from the agent to the principal. This is, however, ruled out by our assumption of limited liability.

The problem on characterizing the best sequentially rational equilibrium for the principal is conceptually similar to the full commitment problem in the previous section. By Lemma 2 in the appendix, the principal can implement all wage and effort paths that satisfy the requirement that $V(t) \geq V^{SR}$ for all t . Hence it is easy to express the maximization problem of the principal in a manner analogous to the previous Section. It is slightly more convenient to cast the analysis in terms of induced effort levels rather than wages.

Consider then the following family $\mathcal{SR}(W)$ of optimization problems:

$$\mathcal{SR}(W) : \max_{\{a(t), W(0), T\}} V(0) \quad \text{subject to}$$

$$\dot{V}(t) = (r_P + a(t))V(t) - a(t)(v - W(t) - c'(a(t))), \quad (15)$$

$$\dot{W}(t) = r_A W(t) - (a(t)c'(a(t)) - c(a(t))). \quad (16)$$

$$V(T) = V^{SR}, W(T) = W. \quad (17)$$

Denote the optimum value of this program by $V_0(W)$. We claim that $V_0(W)$ is decreasing in W . This fact, which is easy to demonstrate, reflects that low continuation values for the agent induce higher effort levels prior to T . Let \underline{W} denote the minimal sequentially rational continuation payoff to the agent. Our main result in this section is that the best sequentially rational payoff to the principal is given by the solution to $\mathcal{SR}(\underline{W})$.

Proposition 7 *The wage path with the highest expected payoff to the principal over all sequentially rational equilibria is given by the solution to $\mathcal{SR}(\underline{W})$ up to time T . Hence the equilibrium wage, action and values all decrease before time T . From T onwards, the continuation payoffs are given by (V^{SR}, \underline{W}) . The equilibrium wages and effort choices*

after T are given by the constants $(\underline{w}, \underline{a})$ that can be solved from:

$$\begin{aligned}\frac{a^{SR} (v - w^{SR})}{r_P + a^{SR}} &= \frac{\underline{a} (v - \underline{w})}{r_P + \underline{a}}, \\ c'(\underline{a}) &= \underline{w} - \underline{W}, \\ \underline{W} &= \frac{\underline{a}\underline{w} - c(\underline{a})}{r_A + \underline{a}} = \frac{\underline{a}c'(\underline{a}) - c(\underline{a})}{r_A}.\end{aligned}$$

Proof. See the appendix. ■

The implied laws of motion up to time T in this best sequentially rational equilibrium satisfy the differential equations derived in the previous section for the full commitment solution. We can therefore determine the dynamics of the best sequentially rational equilibrium using the same phase diagrams. Figure 3 illustrates this for the case $r_P = r_A$. From proposition 3, the full commitment solution path, shown as the dotted line, converges to the origin. The sequentially rational equilibrium path is shown as the solid line. Since the dynamics of this equilibrium are the same (before T) as the full commitment solution, its path cannot cross the full commitment path. We also know that the sequentially rational equilibrium hits the curve $\dot{W} = 0$ at time T . Hence the sequentially rational path must start above the full commitment path, as shown.

6 Project quality

In the analysis so far, the benefit to the principal from a completed project is fixed and verifiable. More generally, we might suppose that the agent is able to affect the quality, and hence the benefit to the principal, of the completed project.

We shall not attempt a general analysis of this issue here. Instead, we outline a variant of our model where project quality has no effect the overall conclusions. Suppose that an agent can affect the probability of project completion by exerting effort, in the same way as in previous sections. But now, the quality of a completed project is drawn at random from a distribution $F(v)$ where $v \geq 0$, which is common knowledge.

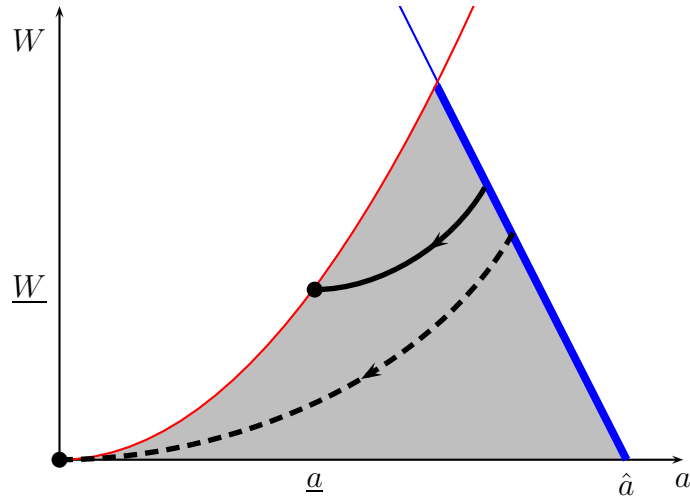


Figure 3: Phase diagram with $r_P = r_A$

At this point, there are two possibilities, which (it will turn out) are equivalent. The first is that the realised project quality cannot be verified. (For example, the agent may be a headhunter and the completed project a candidate for an executive position in the principal's firm. The fit of the candidate for the principal's post may be very difficult to establish to a third-party.) Hence the principal cannot condition payment on the realised project quality.

The second possibility is that the project quality is verifiable. This then raises the possibility that the principal pays only when the realised project quality is at least as great as the current completion wage. But it is easy to see that this cannot be optimal. The agent's expected payment on completing at time t would be $\mathbb{E}[w|v \geq w]$ where the expectation is taken with respect to the quality distribution $F(\cdot)$. The principal gains $\mathbb{E}[v - w|v \geq w]$. But the principal could offer this level of payment unconditionally (and hence present the agent with the same effort incentives); and then accept all completed projects. The principal would gain $\mathbb{E}[v|v \leq w]$ from this.

So, the principal will continue to pay only on project completion. The problem is unaltered by this modification: the principal's benefit v is replaced by an expected benefit—that is all. Hence our previous analysis continues to hold.

The crucial feature of this example is that the agent's effort affects only the arrival

rate of project completion, but not the realised quality. If project quality is affected (stochastically) by effort levels, then the principal will adjust the wage in order to affect both the quality and rate of completion of the project. We leave this and other related issues to further work.

7 Conclusions

We have developed a model of dynamic moral hazard involving project completion which has allowed us to identify clearly the intertemporal incentives involved. We show in the full commitment solution how the principal controls the agent's continuation value over time to provide the optimal dynamic incentives for current effort. The long-run outcome of these dynamics is determined by the relative discount rates of the principal and the agent. The principal never fires the agent—despite the fact that, when the agent is relatively patient, there may be a positive probability that the project is never completed. We also characterise the set of sequentially rational equilibria, in particular identifying the minimal and maximal payoffs of the principal.

The framework that we have developed is very tractable and offers a base from which we intend to explore further dynamic incentives for this type of problem.

Appendix

7.1 Proof of Proposition 1

Start with $w(t) = w^S$ for all t as computed above, and consider a perturbed wage policy where $\hat{w}(s) = w$ for $t \leq s < t + \Delta$ and $\hat{w}(s) = w^S$ for $s \geq t + \Delta$. Then the agent's continuation value is given by W^S and by differentiating the first-order condition (2) with respect to a and w , we have

$$\frac{da(t)}{dw(t)} = \frac{1}{c''(a(t))}.$$

To determine the optimal wage offer for the principal at t , write the value function under the continuation wages at w^S as follows:

$$V(w, V^S) = \max_w \{a(w, W^S) \Delta (v - w) + e^{-r\Delta} (1 - a(w, W^S) \Delta) V^S\}.$$

The derivative with respect to w is then

$$\frac{\partial V(w, V^S)}{\partial w} = \frac{\partial a(w, W^S)}{\partial w} (v - w - V^S) - a(w(t), W^S).$$

At $w = w^S$, we have

$$a'(w^S) (v - w^S - V^S) - a(w(t), W^S) = 0;$$

and so

$$\begin{aligned} \frac{\partial V(w^S, V^S)}{\partial w} &= \left(\frac{\partial a(w^S, W^S)}{\partial w} - a'(w^S) \right) (v - w^S - V^S) \\ &= \frac{W'(w^S)}{c''(a^S)} (v - w^S - V^S) \geq 0. \end{aligned}$$

Hence we conclude that offering a $w > w^S$ for $[t, t + \Delta)$ increases the payoff to the principal. ■

7.2 Proof of Proposition 2

Suppose not i.e., suppose that it is optimal to choose a finite stopping time T . We show that this leads to a contradiction.

Since $W(T)$ can be freely chosen by the principal (subject only to a non-negativity constraint), it must be that $\mu(T) = 0$. Hence $\mu(0) = \mu(T) = 0$ (recalling that $W(0)$ also is freely chosen.) Equation (12) tells us that $\dot{\mu}(0) = -\lambda a(0)$ and $\dot{\mu}(T) = -\lambda a(T)$. Since $a(0)$ and $a(T)$ are non-negative and $\lambda \neq 0$, this means that either $\dot{\mu}(0)$ and $\dot{\mu}(T)$ are both positive or they are both negative (determined by the sign of λ). We analyse fully the former case; the latter case follows an identical argument. So, suppose $\dot{\mu}(0)$ and $\dot{\mu}(T)$ are

both positive (and $\mu(0) = \mu(T) = 0$). Since μ is a continuous function of t , there must be some $t^* \in (0, T)$ such that $\mu(t^*) = 0$, and $\mu(\cdot)$ is decreasing in t at this point. But the latter cannot be the case, since equation (12) implies that $\dot{\mu}(t^*) = -\lambda a(t^*) \geq 0$.

Hence a contradiction has been established, and so a finite stopping time T cannot be optimal. ■

Proof of Proposition 3

The proof uses a phase diagram in (a, W) space. Three aspects need to be analysed: the dynamics of a and W ; and the sign of $V'(\cdot)$ in equilibrium.

The sign of \dot{W} is, from equation (6), determined by the sign of $r_A W - (ac'(a) - c(a))$. This defines an upward-sloping function in (a, W) space, given by

$$W_{\dot{W}=0}(a) \equiv \frac{ac'(a) - c(a)}{r_A}$$

so that for $W > (<)W_{\dot{W}=0}(a)$, $\dot{W} > (<)0$. Note that $W_{\dot{W}=0} = 0$.

We now determine the sign of μ in equilibrium. We first establish that the Hamiltonian $\mathcal{H}(t)$ is zero along the optimal path. Differentiation of $\mathcal{H}(t)$ gives

$$\dot{\mathcal{H}}(t) = -\ddot{V}(t) + \dot{\mu}(t)\dot{W}(t) + \mu(t)\ddot{W}(t).$$

(Here, a double dot denotes a second derivative with respect to time.) After substitution of the expressions for the various terms and simplification, we end up with $\dot{\mathcal{H}}(t) = (r_P + a(t))\mathcal{H}(t)$. There are then two possibilities: either $\mathcal{H}(t) > 0$ or $\mathcal{H}(t) = 0$ for all t . If the former, then $\mathcal{H}(t)$ grows at least exponentially over time; this clearly cannot be the case on any feasible path. Therefore $\mathcal{H}(t) = 0$ for all t .

We can therefore write \dot{V} as $\mu\dot{W}$. Now use equations (7) and (10) to give

$$(r_A W - ac'(a) + c(a) + (a + r_P)ac''(a))\mu = r_P(v - W - c'(a)) - (a + r_P)ac''(a). \quad (18)$$

Consider the left-hand side of this equation. From assumption 1, $r_A W - ac'(a) + c(a) + (a +$

$r_P)ac''(a) \geq 0$ for all non-negative values of a and W . Hence the sign of μ is determined by the sign of $r_P(v - W - c'(a)) - (a + r_P)ac''(a)$. This defines a function in (a, W) space, given by

$$W_{\mu=0}(a) \equiv v - c'(a) - \left(\frac{a + r_P}{r_P}\right)ac''(a).$$

This is a downward-sloping function, with an intercept $W_{\mu=0}(0) = v$; it hits the horizontal axis at an effort level \hat{a} strictly less than the myopic level a^M (defined by $v - c'(a^M) - a^M c''(a^M) = 0$). For values of (a, W) lying below (above) this function, μ is positive (negative); along the function, $\mu = 0$. The function $W_{\dot{W}=0}(a)$ is, therefore, split into two portions by the function $W_{\mu=0}(a)$; call the intersection point of the two functions (a^*, W^*) .

Now consider the dynamics of a , determined by equation (14). When $\mu = 0$ (in particular, at the optimal initial choice of W), the term on the left-hand side, $2c''(a(t)) + a(t)c'''(a(t))$, is non-negative (using assumption 1). The right-hand side is equal to

$$-(r_A W - (a(t)c'(a(t)) - c(a(t))) + a^2 c''(a)),$$

which by assumption 1 is negative for all non-negative values of a and W . Hence $\dot{a} \leq 0$ at the optimal initial choice of W .

The function defined by $\dot{a} = 0$ is

$$W_{\dot{a}=0}(a) \equiv \frac{1}{r_A} \left(ac'(a) - c(a) - ac''(a) \left(a + (r_P - r_A) \frac{\mu}{\mu + 1} \right) \right).$$

Note that when $\mu = 0$,

$$W_{\dot{a}=0}(a) \equiv \frac{ac'(a) - c(a) - a^2 c''(a)}{r_A} \leq 0$$

from assumption 1. Hence the function $W_{\dot{a}=0}(a)$ crosses the function $W_{\mu=0}(a)$ at a point below the horizontal axis. If $r_P = r_A$, then $W_{\dot{a}=0}(a) \leq 0$ for all $a \in [0, \hat{a}]$. If $r_P < r_A$,

then $W_{\dot{a}=0}(a) > W_{\dot{W}=0}(a)$ for sufficiently small a . Since $W_{\dot{a}=0}(a)$ is continuous in a , the $W_{\dot{a}=0}(a)$ curve must therefore cross the $W_{\dot{W}=0}(a)$ at a value of a that is strictly less than a^* .

We can now determine the dynamics of a and W . The region of particular interest for the analysis is defined as follows. Let

$$\mathcal{W}(a) \equiv \{W \in \mathbb{R}_+ | W \leq W_{\dot{W}=0}(a) \text{ and } W \leq W_{\mu=0}(a) \text{ and } W \geq W_{\dot{a}=0}(a)\}$$

for $a \in [0, \hat{a}]$. Let

$$\mathcal{D} \equiv \bigcup_{a \in [0, \hat{a}]} \mathcal{W}(a).$$

(The region \mathcal{D} is illustrated as the shaded regions in figures 1 and 2.) For $(a, W) \in \mathcal{D}$, both \dot{a} and \dot{W} are non-positive. If $r_P = r_A$, then \mathcal{D} is defined as the (lower) area between the curves $W_{\dot{W}=0}(a)$ and $W_{\mu=0}(a)$. If $r_P < r_A$, then \mathcal{D} is further defined by the curve $W_{\dot{a}=0}(a)$.

An initial choice of W on the portion of the $W_{\mu=0}(a)$ curve above the $W_{\dot{W}=0}(a)$ cannot be optimal. The reason is that the dynamics from this point involve $\dot{W} \geq 0$. Hence any path from such a point cannot converge to a steady state, and by transversality cannot be optimal.

Hence the optimal initial choice of W must lie on the portion of the $W_{V'=0}(a)$ curve below the $W_{\dot{W}=0}(a)$. (Note that this must involve an initial effort level less than \hat{a} , and hence less than the myopic level a^M .) The initial dynamics from such a point are $\dot{W} \leq 0$ and $\dot{a} \leq 0$. The resulting path lies in the region \mathcal{D} , and hence $\dot{W} \leq 0$ and $\dot{a} \leq 0$ along all parts of an optimal path. The dynamics of $w(t)$ then follows from equation (5).

If $r_P = r_A$, then assumption 1 ensures that the function $W_{\dot{a}=0}(a)$ is negative for all values of $a \in [0, \hat{a}]$. Hence the only feasible steady state is $a = W = 0$. If $r_P < r_A$, then there is a second steady state with strictly positive $a \in (0, a^*)$ and $W \in (0, W^*)$; the optimal path converges to this steady state. (Note that the system cannot converge to a point on the horizontal axis. At such a point, the agent's effort is strictly positive but her value is 0. But this cannot be optimal for the agent, who could generate a strictly

positive value by reducing her effort.) ■

Proof of Proposition 4

The proposition is immediate when the agent is relatively impatient ($r_P < r_A$), since his equilibrium effort in this case is always strictly positive. To prove the proposition when the agent is equally patient ($r_P = r_A$), we consider the behaviour of the system for small values of a and W .

From equation (18), as $(a, W) \rightarrow (0, 0)$,

$$\mu \rightarrow \frac{r_P(v - W - c'(a)) - (a + r_P)ac''(a)}{r_A W + (a + r_P)ac''(a)}$$

and so becomes unbounded as the system approaches the steady state. Using this fact, the dynamics of the agent's effort, from equation (14), tends towards

$$\begin{aligned} & [c''(a(t)) + a(t)c'''(a(t))]\dot{a}(t) \\ & = -(r_A W - (a(t)c'(a(t)) - c(a(t))) + a^2 c''(a)) + a^2(t)c''(a(t))(r_P - r_A). \end{aligned} \quad (19)$$

The expression in the square brackets on the left-hand side of this equation is of order (big-O) $c''(a)$ in a . The right-hand side of the equation is of order (big-O) $c(a)$ in a and is linear in W .

Next, let the path followed by W in (a, W) space be denoted $W(a)$. Given the dynamics described by \dot{a} and \dot{W} , this path is smooth. Consider a Taylor expansion of this path close to the steady state $(0, 0)$:

$$W(a) = W(0) + aW'(0) + a^2W''(0) + \dots$$

Obviously $W(0) = 0$; and since $W(a) \geq 0$ for all a , it must therefore be that $W'(0) \geq 0$. But since $\dot{W} \leq 0$ (proposition 3), it must also be that

$$W(a) \leq W_{\dot{W}=0}(a) \equiv \frac{ac'(a) - c(a)}{r_A}.$$

Therefore

$$W'(0) \leq W'_{\dot{W}=0}(a) = \left. \frac{ac''(a)}{r_A} \right|_{a=0} = 0.$$

Therefore $W'(0) = 0$, and the Taylor expansion of $W(a)$ becomes

$$W(a) = a^2 W''(0) + \dots$$

That is, on the optimal path, $W(a) = O(a^2)$.

Using this fact in equation (19), we see that the left-hand side is of order (big-O) $c''(a)$ in a , while the right-hand side is of order (big-O) a^2 . Let $\beta(a) \equiv a^2/c''(a)$ for $a > 0$. (For example, with quadratic costs $c(a) = \gamma a^2$ with $\gamma > 0$, $\beta(a) = a^2/2\gamma$.) Hence

$$\dot{a} = O(\beta(a)) \text{ as } (a, W) \rightarrow (0, 0).$$

Whether the integral $\int_0^\infty a(t)dt$ of the agent's effort is bounded above by a finite constant then depends on the order of $\beta(a)$. It is straightforward to show that if $\beta(a)$ is of order a^2 or higher, then the rate of decrease is sufficiently slow that the integral is unbounded. But if $\beta(a)$ is of lower order than a^2 , then the rate of decrease is such that the integral is finite. ■

The Dynamic Contracting Game

At the beginning of each period t , the principal promises to pay w_t^s if the project is completed and w_t^f if it remains incomplete. All past promises by the principal are observable but as in the previous sections, we maintain the assumption that the agent's effort choice is unobservable. Since the game ends once the project is completed (and this is verifiable), the only relevant histories in the game are those conditional on the project not being completed. We denote a history up to period t by:

$$h^t = (w_0^s, w_0^f, \dots, w_{t-1}^s, w_{t-1}^f)$$

The set of histories up to period t is denoted by H^t . The set of all histories of all lengths is $H = \cup_t H^t$. A wage setting strategy \mathbf{w} for the principal is a pair of functions $(\mathbf{w}^s, \mathbf{w}^f)$ where

$$\mathbf{w}^s : H \rightarrow \mathbb{R},$$

$$\mathbf{w}^f : H \rightarrow \mathbb{R}.$$

A continuation wage setting strategy after history h^t is denoted by $\mathbf{w}|_{h^t}$. Since the agent chooses her effort in period t only after observing the promised wages, her strategy is denoted by \mathbf{a} where⁷

$$\mathbf{a} : H \rightarrow \mathbb{R}.$$

A continuation strategy for the agent after history h^t is denoted by $\mathbf{a}|_{h^t}$.

Strategies induce payoffs to the principal and to the agent in a manner similar to the previous section. Letting τ denote the random time at which the project is completed, we have the expected payoffs for the principal and for the agent as:

$$\begin{aligned} V(\mathbf{w}, \mathbf{a}) &= E \left[(1 - r\Delta)^\tau (v - \mathbf{w}^s(h^\tau)) - \sum_{j=0}^{\tau-1} (1 - r\Delta)^j \mathbf{w}^f(h^j) \right] \\ &= \sum_{i=0}^{\infty} \Pr\{\tau = i\} \left[(1 - r\Delta)^\tau (v - \mathbf{w}^s(h^\tau)) - \sum_{j=0}^{i-1} (1 - r\Delta)^j \mathbf{w}^f(h^j) \right], \\ W(\mathbf{w}, \mathbf{a}) &= E \left[(1 - r\Delta)^\tau \mathbf{w}^s(h^\tau) - \sum_{j=0}^{\tau} (1 - r\Delta)^j c(\mathbf{a}(h^j)) + \sum_{j=0}^{\tau-1} (1 - r\Delta)^j \mathbf{w}^f(h^j) \right] \\ &= \sum_{i=0}^{\infty} \Pr\{\tau = i\} \left[(1 - r\Delta)^i \mathbf{w}^s(h^i) - \sum_{j=0}^i (1 - r\Delta)^j c(\mathbf{a}(h^j)) + \sum_{j=0}^{i-1} (1 - r\Delta)^j \mathbf{w}^f(h^j) \right]. \end{aligned}$$

Letting $\Delta \rightarrow 0$, we recover payoffs analogous to the continuous-time payoffs used in the previous sections.

⁷Since the stage game is in extensive form, and the principal moves first, the relevant history for the agent in period t also contains w_t and is thus h^{t+1} .

Proof of Proposition 5

We start by showing that in any stationary equilibrium $\mathbf{w}^f(h^t) = w^f = 0$ for all h^t . Suppose not. Then $w^f > 0$. If $w^s \leq w^f$, then the optimal effort choice is 0 and the equilibrium payoff to the principal is negative and setting $w^f(h^t) = w^s = 0$ is a profitable deviation after any h^t . Suppose next that $w^s > w^f > 0$. A deviation to $\hat{w}^s(h^t) = w^s - w^f$, $\hat{w}^f(h^t) = 0$ leaves the incentives for optimal effort choice in period t unchanged since by assumption the future wage offers by the principal do not depend on the current ones in a stationary equilibrium. This deviation reduces the wage payments of the principal by $w^f > 0$ and hence is clearly profitable.

We compute next a wage offer $w^s = w^{SR} > 0$ and an effort choice function a that constitute a stationary equilibrium for our game. We start by computing the equilibrium expected payoff W^{SR} to the agent. By standard arguments, we have for $\Delta \rightarrow 0$:

$$W^{SR} = \max_a \frac{aw^{SR} - c(a)}{r_A + a}. \quad (20)$$

By solving the first-order condition for a and substituting, we get:

$$r_A W^{SR} = a^{SR} c'(a^{SR}) - c(a^{SR}),$$

where a^{SR} solves problem (20.) Therefore we have after substituting:

$$w^{SR} = c'(a^{SR}) + \frac{a^{SR} c'(a^{SR}) - c(a^{SR})}{r_A}. \quad (21)$$

Consider next a current (possibly deviating) wage offer of $w^s(h^t) = w, w^f(h^t) = 0$.⁸ The agent's best response to this offer is given by a solution to the following problem:

$$\max_a \{aw\Delta - c(a)\Delta + (1 - r_A\Delta)(1 - a\Delta)W^{SR}\}.$$

⁸By the same argument as above, any deviation where $w_t^f > 0$ is dominated by a deviation where $w_t^f = 0$, but the incentives are unchanged.

The first-order condition for the problem is given by:

$$(w - W^{SR}) = c'(a).$$

Denote the solution to this problem by $a(w, 0)$. By the implicit function theorem,

$$\frac{da}{dw} = \frac{1}{c''(a(w, 0))}. \quad (22)$$

The expected payoff to the principal from this wage offer is

$$a(w, 0) \Delta(v - w) + (1 - r_P \Delta)(1 - a(w, 0) \Delta) V^{SR}, \quad (23)$$

where V^{SR} is the expected payoff to the principal along the path where $w_t^s = w^{SR} > 0$ for all t . For $\Delta \rightarrow 0$, the first order condition for this problem is:

$$(v - w - V^{SR}) \frac{da}{dw} - a(w, 0) = 0.$$

Substituting from equation (22), we have:

$$(v - w - V^{SR}) - a(w, 0) c''(a(w, 0)) = 0.$$

At a stationary equilibrium, we must have the first order condition satisfied at $w = w^{SR}$.

Therefore

$$w^{SR} = v - \frac{a^{SR}(a^{SR} + r_P)}{r_P} c''(a^{SR}). \quad (24)$$

In a stationary equilibrium, equations (21) and (24) must hold simultaneously. By our assumptions, the former gives w^{SR} as an increasing function of a^{SR} while the latter gives a decreasing function. At $a^{SR} = 0$, the former function starts above the latter while in the limit as $a^{SR} \rightarrow \infty$ the latter decreases without bound. Therefore the values of these functions coincide at exactly one point. ■

Proof of Proposition 6

The proof consists of a number of steps. The first Lemma establishes the existence of a smallest equilibrium payoff for the principal.

Lemma 1 *There exists a number \underline{V} such that*

$$\underline{V} = \min\{V(\mathbf{w}, \mathbf{a}) \mid (\mathbf{w}, \mathbf{a}) \in \mathcal{E}\}.$$

Proof. For any fixed $\Delta > 0$, there is a \bar{w} such that any wage offer strategy with $w_t > \bar{w}$ is strictly dominated for the principal. Under our assumption of limited liability, the agent cannot make any transfers to the principal. Hence the principal's relevant choice set is compact. By the strict convexity of the agent's effort cost, the same is true for the agent's relevant action set.

Since any path is an element in a countable Cartesian product of these compact stage game action sets, the set of paths is compact in the product topology (Tychonoff's theorem). Furthermore, the stage game payoffs of the principal and the agent are continuous in their actions. As a result, we can apply the same reasoning as in Proposition 2 of [Abreu \(1988\)](#) to conclude the existence of a sequentially rational equilibrium that minimizes the payoff to the principal. ■

The second step shows that all wage offer paths that induce an expected payoff of at least V^{SR} on all continuation paths can be supported in sequentially rational equilibrium. To be more precise, we define a path of wages from t onwards $w|_t := \{w_s\}_{s=t}^{\infty}$ to be a sequence of wage offers w_s for all future periods s conditional on success or failure. Similarly, we define a path of effort choices $a|_t := \{a_s\}_{s=t}^{\infty}$ for the agent.

Lemma 2 *Suppose that for all t ,*

$$V(w|_t, a|_t) \geq V^{SR} = \underline{V},$$

and that $a|_t$ is a best response to $w|_t$. Then $w|_t$ is the equilibrium wage offer path of some sequentially rational equilibrium.

Proof. Let $(\mathbf{w}^*, \mathbf{a}^*)$ be any sequentially rational equilibrium for which

$$V(\mathbf{w}^*, \mathbf{a}^*) = \underline{V}.$$

Then also $(\mathbf{w}^*|_{h_1=w}, \mathbf{a}^*)$ is a sequential equilibrium for any arbitrary w . Since $(\mathbf{w}^*, \mathbf{a}^*)$ is an equilibrium with payoff V^{SR} to the principal, the payoff induced by $(w, \mathbf{w}^*|_{h_1=w})$ cannot exceed V^{SR} .

By assumption, the payoff from wage path $w|_t$ is at least V^{SR} for all t . Consider the following strategies. As long as the principal has offered wages on $w|_s$, both players continue to choose actions on the path. If the principal deviates to $w' \neq w_t$ at any t , then play reverts to $(\mathbf{w}^*|_{h_1=w}, \mathbf{a}^*)$. Therefore the expected payoff to the principal is always at least V^{SR} on the path and no larger than V^{SR} for any other choice. By construction, the agent is always best-responding to the principal's strategy. ■

Lemma 3 *Let $(\mathbf{w}, \mathbf{a}) \in \mathcal{E}$, such that $V(\mathbf{w}, \mathbf{a}) > V^{SR}$ and $W(\mathbf{w}, \mathbf{a}) = W$. Then there is a $(\mathbf{w}', \mathbf{a}') \in \mathcal{E}$ such that*

$$V(\mathbf{w}', \mathbf{a}') = \underline{V}, W(\mathbf{w}', \mathbf{a}') = W + V(\mathbf{w}, \mathbf{a}) - \underline{V} > W.$$

Proof. Write $\mathbf{w} = (w, \widehat{\mathbf{w}})$ and let

$$V(\mathbf{w}, \mathbf{a}) - \underline{V} = \varepsilon > 0.$$

Consider then a deviation to

$$\begin{aligned} \widetilde{w}^s(h^0) &= \widehat{w}^s(h^0) + \varepsilon, \\ \widetilde{w}^f(h^0) &= \widehat{w}^f(h^0) + \varepsilon. \end{aligned} \tag{25}$$

Since the wages are increased, the new wages are also non-negative, and furthermore, the incentives of the agent are unchanged in the continuation game.

By Lemma 2, $((\tilde{w}, \hat{\mathbf{w}}), \mathbf{a})$ is a sequentially rational equilibrium. ■

Corollary 1 *Let $(\mathbf{w}^*, \mathbf{a}^*) \in \mathcal{E}$ be such that $V(\mathbf{w}^*, \mathbf{a}^*) = \underline{V}$. Then $V(\mathbf{w}^*|_{h^t}, \mathbf{a}^*|_{h^t}) = \underline{V}$ for all h^t on the equilibrium path of $(\mathbf{w}^*, \mathbf{a}^*)$.*

Proof. Suppose to the contrary and apply the same construction as in equation (25) to obtain a lower equilibrium payoff to the principal after some history on the equilibrium path and therefore a lower ex ante payoff. ■

Finally, consider the set of equilibria that induce payoff \underline{V} to the principal after all histories on the equilibrium path. Let $(\mathbf{w}^{\min}, \mathbf{a}^{\min})$ solve

$$\begin{aligned} & \max_{(\mathbf{w}, \mathbf{a}) \in \mathcal{E}} W(\mathbf{w}, \mathbf{a}) \\ & \text{s.t. } V(\mathbf{w}, \mathbf{a}) = \underline{V}. \end{aligned}$$

The final Lemma shows that $(\mathbf{w}^{\min}, \mathbf{a}^{\min})$ can be used as the continuation path following all initial wage offers in any equilibrium inducing the minimal payoff of \underline{V} to the principal.

Lemma 4 *For all $w \in \mathbb{R}$,*

$$V((w, \mathbf{w}^{\min}), \mathbf{a}^{\min}) \leq V(\mathbf{w}, \mathbf{a}) \text{ for all } (\mathbf{w}, \mathbf{a}) \in \mathcal{E}.$$

Proof. For a fixed $w_t = w$, the current effort choice a_t is decreasing in $W(\mathbf{w}, \mathbf{a})$. For fixed w and a continuation payoff $V(\mathbf{w}, \mathbf{a})$ for the principal, the equilibrium value to the principal is linear in the current effort a_t :

$$V((w, \mathbf{w}), \mathbf{a}) = a_t \Delta (v - w_t) + (1 - a_t \Delta) (1 - r \Delta) V(\mathbf{w}, \mathbf{a}).$$

Therefore it is increasing in a_t whenever $v - w_t \geq (1 - r\Delta)V$. By Lemma 3, $W(\mathbf{w}, \mathbf{a})$ is maximized where $V(\mathbf{w}, \mathbf{a})$ is minimized over $(\mathbf{w}, \mathbf{a}) \in \mathcal{E}$. Hence $(\mathbf{w}^{\min}, \mathbf{a}^{\min})$ solves also

$$\begin{aligned} \min_{(\mathbf{w}, \mathbf{a}) \in \mathcal{E}} V((w, \mathbf{w}), \mathbf{a}) \\ \text{s.t. } V(\mathbf{w}, \mathbf{a}) = \underline{V}. \end{aligned}$$

If $v - w_t < (1 - r\Delta)V_{t+1}$, then $a_t\Delta(v - w_t) + (1 - a_t\Delta)(1 - r\Delta)V_{t+1} < V_{t+1} = \underline{V}$ by Corollary 1. ■

With this Lemma, the proof of the proposition is complete since $(\mathbf{w}^{\min}, \mathbf{a}^{\min})$ is a stationary equilibrium by construction. ■

Proof of Proposition 7

By Corollary 1, we know that the expected payoff to the principal in any equilibrium inducing payoff \underline{V} is constant. We start by showing that in the solution to the problem $\mathcal{SW}(\underline{V})$, the expected payoff to the agent (conditional on no success) is also constant.

Suppose to the contrary. Then any equilibrium inducing the minimal payoff to the agent must have histories h^t and h^{t+1} on the equilibrium path with the property that

$$W(\mathbf{w}|_{h^{t+1}}, \mathbf{a}|_{h^{t+1}}) > W(\mathbf{w}|_{h^t}, \mathbf{a}|_{h^t}).$$

Denote the sequence of continuation wage offers on such an equilibrium path by $\{w^s\}_{s=t}^{\infty}$. Consider an alternative wage offer sequence $\{\tilde{w}^s\}_{s=t}^{\infty}$, where

$$\tilde{w}^{t+k} = w^{t+k-1} \text{ for } k > 0.$$

In other words, the modified sequence $\{\tilde{w}^s\}_{s=t}^{\infty}$ takes the original equilibrium continuation path after h^t as the continuation path after (h^t, \tilde{w}^t) .

By Lemma 2, the modified sequence $\{\tilde{w}^s\}_{s=t+1}^{\infty}$ can be supported as the equilibrium path of a sequentially rational equilibrium since $V(\mathbf{w}|_{h^{t+1}}, \mathbf{a}|_{h^{t+1}}) = \underline{V}$. We need to show

that $\{\tilde{w}^s\}_{s=t}^\infty$ also can be supported as part of an equilibrium for a suitable choice of \tilde{w}^t .

By construction,

$$W(\tilde{w}|_{t+1}, \tilde{a}|_{t+1}) < W(\mathbf{w}|_{h^{t+1}}, \mathbf{a}|_{h^{t+1}}),$$

where $\tilde{a}|_{t+1} = \{\tilde{a}\}_{s=t+1}^\infty$ is the sequence of optimal effort choices by the agent when the future wage sequence is given by $\{\tilde{w}^s\}_{s=t}^\infty$. Since the agent's current effort is decreasing in the continuation value we have at $\tilde{w}^t = w^t$,

$$\tilde{a}^t > a^t.$$

Clearly,

$$\tilde{a}^s = a^s \text{ for all } s > t.$$

Continuing with the assumption that $\tilde{w}^t = w^t$, we have:

$$W((w^t, \tilde{w}|_{t+1}), (\tilde{a}^t, \tilde{a}|_{t+1})) < W(\mathbf{w}|_{h^t}, \mathbf{a}|_{h^t})$$

and

$$V((w^t, \tilde{w}|_{t+1}), (\tilde{a}^t, \tilde{a}|_{t+1})) > V(\mathbf{w}|_{h^t}, \mathbf{a}|_{h^t}) = \underline{V}$$

since the current payoff to the principal is increasing in \tilde{a}^t .

The last step shows how to modify \tilde{w}^t in such a manner that $V(\tilde{w}|_t, \tilde{a}|_t) = \underline{V}$, and the expected payoff to the agent is further decreased. Fixing the continuation value to the agent W , the current period expected payoff to the principal is continuous in \tilde{w}^t . This payoff is zero at $\tilde{w}^t = 0$ and $\tilde{w}^t = v$. By continuity, there must be a $\tilde{w}^t < w^t$ such that

$$V(\tilde{w}|_t, \tilde{a}|_t) = \underline{V}.$$

Since the current payoff to the agent is increasing in \tilde{w}^t , the payoff to the agent decreases when changing from w^t to \tilde{w}^t . ■

References

- ABREU, D. (1988): “On the Theory of Infinitely Repeated Games with Discounting,” *Econometrica*, 56(2), 383–396.
- BERGEMANN, D., AND U. HEGE (2005): “The financing of innovation: learning and stopping,” *RAND Journal of Economics*, 36(4), 719–752.
- FERNANDEZ-MATEO, I. (2003): “How Free Are Free agents? Relationships and Wages in a Triadic Labor Market,” Available at http://www.london.edu/assets/documents/Isabel_paper.pdf.
- FINLAY, W., AND J. COVERDILL (2000): “Risk, Opportunism, and Structural Holes: How Headhunters Manage Clients and Earn Fees,” *Work and Occupations*, 27, 377–405.
- HOPENHAYN, H., AND J. P. NICOLINI (1997): “Optimal Unemployment Insurance,” *Journal of Political Economy*, 105, 412–438.
- KAMIHIGASHI, T. (2001): “Necessity of Transversality Conditions for Infinite Horizon Problems,” *Econometrica*, 69(4), 995–1012.
- LAFFONT, J.-J., AND D. MARTIMORT (2002): *The Theory of Incentives: the Principal-Agent Model*. Princeton University Press.
- MALCOMSON, J., AND F. SPINNEWYN (1988): “The Multi-Period Principal-Agent Problem,” *Review of Economic Studies*, 55, 391–408.
- MERLO, A., AND F. ORTALO-MAGNÉ (2004): “Bargaining over Residential Properties: Evidence from England,” *Journal of Urban Economics*, 56, 192–216.
- MERLO, A., F. ORTALO-MAGNÉ, AND J. RUST (2006): “Bargaining and Price Determination in the Residential Real Estate Market,” Available at http://gemini.econ.umd.edu/jrust/research/nsf_pro_rev.pdf.

- OFFICE OF FAIR TRADING (2004): “Estate Agency Market in England and Wales,” Discussion Paper OFT693, Office of Fair Trading.
- PHARMAFOCUS (2007): “Finding top people for the top job,” Internet Publication, http://www.pharmafocus.com/cda/focusH/1,2109,22-0-0-0-focus_feature_detail-0-491515,00.html.
- PHELAN, C., AND R. TOWNSEND (1991): “Computing Multi-Period, Information-Constrained Optima,” *Review of Economic Studies*, 58, 853–881.
- SANNIKOV, Y. (2007): “Games with Imperfectly Observable Actions in Continuous Time,” *Econometrica*, 75(5), 1285–1329.
- (2008): “A Continuous-Time Version of the Principal-Agent Problem,” *Review of Economic Studies*, 75(3), 957–984.
- SHAVELL, S., AND L. WEISS (1979): “The Optimal Payment of Unemployment Insurance Benefits over Time,” *Journal of Political Economy*, 87(6), 1437–1362.
- SPEAR, S., AND S. SRIVASTAVA (1987): “On Repeated Moral Hazard with Discounting,” *Review of Economic Studies*, 54, 599–617.
- WILLAMS, N. (2006): “On Dynamic Principal-Agent Problems in Continuous Time,” Available at <http://www.princeton.edu/~noahw/pa1.pdf>.