Labor Economics I

FDPE Spring 2015

Marko Terviö (Aalto University)

The Ben-Porath Model

The Ben-Porath Model

Yoram Ben-Porath: The Production of Human Capital and the Life Cycle of Earnings (JPE 1967).

Model of an individual lifetime decision of how to divide time between work and learning.
What is the optimal time profile of human capital investment and earnings?

- Key features
  - Finite career with a fixed retirement time: $t \in [0,T]$
  - Earning potential at time t depends on the level of human capital $H(t)$
  - Building up human capital requires time and money
  - Fixed leisure/labor divide: one unit of non-leisure time per "period"
- Choices at $t \in [0,T]$
  - What fraction of productive time to devote to human capital accumulation
    $s(t) \in [0,1]$
  - How much money to spend on human capital
    $M(t) \geq 0$.

The choice of $M(t)$ is less interesting and could be ignored. (Footnote 5 in the paper is odd!)

Model setup

Here with modernized notation and terminology

- Maximizing PV of lifetime utility = maximizing PV of lifetime consumption, because
  - Perfect capital market: no credit constraints
  - Fixed “time budget constraint” with no interactions with leisure in the utility
  - Exponential discounting

Present value of lifetime consumption

$$W_0 = \int_0^T e^{-\mathrm{rt}}(wH(t)(1 - s(t)) - M(t))\,\mathrm{d}t \tag{1}$$

Evolution of human capital

$$H'(t) = A(s(t)H(t))^{\alpha} M(t)^{\beta} - \delta \mathrm{H}(t) \tag{2}$$

Choice variables

s	share of time devoted to investment in human capital
M	monetary spending on human capital

State variable

H	human capital
$H(0)$	endowment

Parameters

r	discount rate
w	wage rate per units of human capital
δ	depreciation rate of human capital
T	length of career
A,α,β	parameters of a human capital production function, $\alpha+\beta \leq 1$

The economic problem

Maximize W_0, choosing $s(t) \in [0,1]$ and $M(t) \geq 0$, $t \in [0, T]$.

Solution method

Set up the present value Hamiltonian

$$\mathcal{H} = e^{-rt}\left(wH(t)\,(1 - s\,(t)) - M\,(t)\right) + \lambda(t)\left(A(s(t)\,H(t))^{\alpha}\,M(t)^{\beta} - \delta \mathrm{H}(t)\right) \qquad (3)$$

Costate variable λ is the shadow value of investment

Recall the Maximum Principle (aka Pontryagin’s maximum principle).
Maximization requires satisfying these first-order conditions:

$$\frac{\partial \mathcal{H}}{\partial s} = 0 \implies$$

$$e^{-rt}\,w\,H(t) = \lambda(t)\,\mathrm{A}\alpha\,s(t)^{\alpha-1}\,H(t)^{\alpha}\,M(t)^{\beta} \qquad (4)$$

$$\frac{\partial \mathcal{H}}{\partial M} = 0 \implies$$

$$e^{-rt} = \lambda(t)\,\mathrm{A}\beta\,s(t)^{\alpha}\,H(t)^{\alpha}\,M(t)^{\beta-1} \qquad (5)$$

$$\frac{\partial \mathcal{H}}{\partial H} = -\lambda'\,(t) \implies$$

$$e^{-rt}\,w(1 - s(t)) + \lambda(t)\left[\alpha\,A\,s(t)^{\alpha}\,H(t)^{\alpha-1}\,M(t)^{\beta} - \delta\right] = -\lambda'\,(t)$$

Terminal condition

- Human capital cannot be sold and has no “scrap value”
- It is not in general optimal to let H run down to zero in the end. Endogenous $H(T) > 0$.
- Shadow value of H declines to zero when time runs out: $\lambda(T) = 0$.
  - Investment in the last instant of the career is useless: $H'(T) = 0$ is optimal, so $s(T) = 0$, $M(T) = 0$.

Solution steps

Relation of money and time spending

What is the cheapest way to combine time and money to produce units of *H*?

Optimal money investment *M(t)* is mechanically related to the optimal "human capital investment" *s(t)H(t)*.

Production function is Cobb-Douglas, so expenditure shares are proportional to coefficients α and β. "Prices" are *w* for human capital and 1 for money.

$$\frac{w\, s(t)\, H(t)}{M(t)} = \frac{\alpha}{\beta} \Longrightarrow$$

$$M(t) = \frac{w\, \beta}{\alpha}\, s(t)\, H(t) \tag{6}$$

This holds if interior solution $s(t) \in (0,1)$.

Equivalently, solve both (4) and (5) for $\lambda(t)$, then solve for $M(t)$.

Reduce the choice variable FOC

With just one choice variable the problem gets simpler

$$\frac{\partial \mathcal{H}}{\partial s} = 0$$

$$\Longrightarrow \; e^{-\mathrm{rt}}\, w\, H(t) = \lambda(t)\, \mathrm{A}\alpha\, s(t)^{\alpha-1}\, H(t)^{\alpha}\left[\frac{w\,\beta}{\alpha}\, s(t)\, H(t)\right]^{\beta}$$

$$\Longleftrightarrow \; e^{-\mathrm{rt}}\, s(t) \; = \; \lambda(t)\, A_1 s(t)^{\alpha+\beta}\, H(t)^{\alpha+\beta-1} \tag{7}$$

where $A_1 := \; A\left(\frac{\alpha}{w}\right)^{1-\beta} \beta^{\beta}$

Reduce the state variable FOC

$$\frac{\partial \mathcal{H}}{\partial H} = -\lambda'(t)$$

$$\Longrightarrow \quad e^{-rt}\, w(1 - s(t)) + \lambda(t)\Big[\alpha\, A\, s(t)^{\alpha}\, H(t)^{\alpha-1}\Big[\frac{w\,\beta}{\alpha}\, s(t)\, H(t)\Big]^{\beta} - \delta\Big] = -\lambda'(t)$$

$$\Longleftrightarrow \quad e^{-rt}\, w - e^{-rt}\, s(t)\, w + \lambda(t)\big[w\, A_{1} s(t)^{\alpha+\beta}\, H(t)^{\alpha+\beta-1} - \delta\big] = -\lambda'(t) \tag{8}$$

$$\Longleftrightarrow \quad e^{-rt}\, w - \delta\, \lambda(t) = -\lambda'(t) \tag{9}$$

where (7) was used to eliminate *s(t)* and *H(t)* from $\frac{\partial \mathcal{H}}{\partial H}$ in (8)

Eliminating the costate variable

Evolution of the shadow value of investment follows a simple ODE

$\lambda'(t) + \delta\lambda(t) - e^{-rt} w = 0$

Solution is

$$\lambda(t) = \frac{e^{-rt}}{r+\delta} w + e^{\delta t} C_1$$

Terminal condition $\lambda(T) = 0$ pins down the constant C_1

$$\lambda(t) = \frac{e^{-rt}}{r+\delta} w + e^{\delta t}\left(-\frac{e^{rT-\delta T}}{r+\delta} w\right) = \frac{w}{r+\delta}\left(e^{-rt} - e^{\delta t-(\delta+r)T}\right) \tag{10}$$

$$\Longrightarrow \lambda(t) = \frac{w}{r+\delta} e^{-rt}\left(1 - e^{-(\delta+r)(T-t)}\right) \tag{11}$$

Optimal policy

Insert $\lambda(t)$ from (11) to $\frac{\partial \mathcal{H}}{\partial s} = 0$ from (7) and then solve for $s(t)$

$$s(t) = \lambda(t)\, e^{rt} A_1 s(t)^{\alpha+\beta} H(t)^{\alpha+\beta-1} \qquad \Longrightarrow$$

$$s(t) = \frac{w}{r+\delta}\left(1 - e^{-(\delta+r)(T-t)}\right) A_1 s(t)^{\alpha+\beta} H(t)^{\alpha+\beta-1} \qquad \Longleftrightarrow$$

$$s(t) = \frac{1}{H(t)}\left(A_1 \frac{w}{r+\delta}\left(1 - e^{-(\delta+r)(t-T)}\right)\right)^{\frac{1}{1-\alpha-\beta}}$$

This solution is only valid when $s(t) \in (0,1)$.

But $s(t) = 1$ can be optimal for some $t \in [0, t^*]$, “specialization phase.”

Empirical earnings profiles

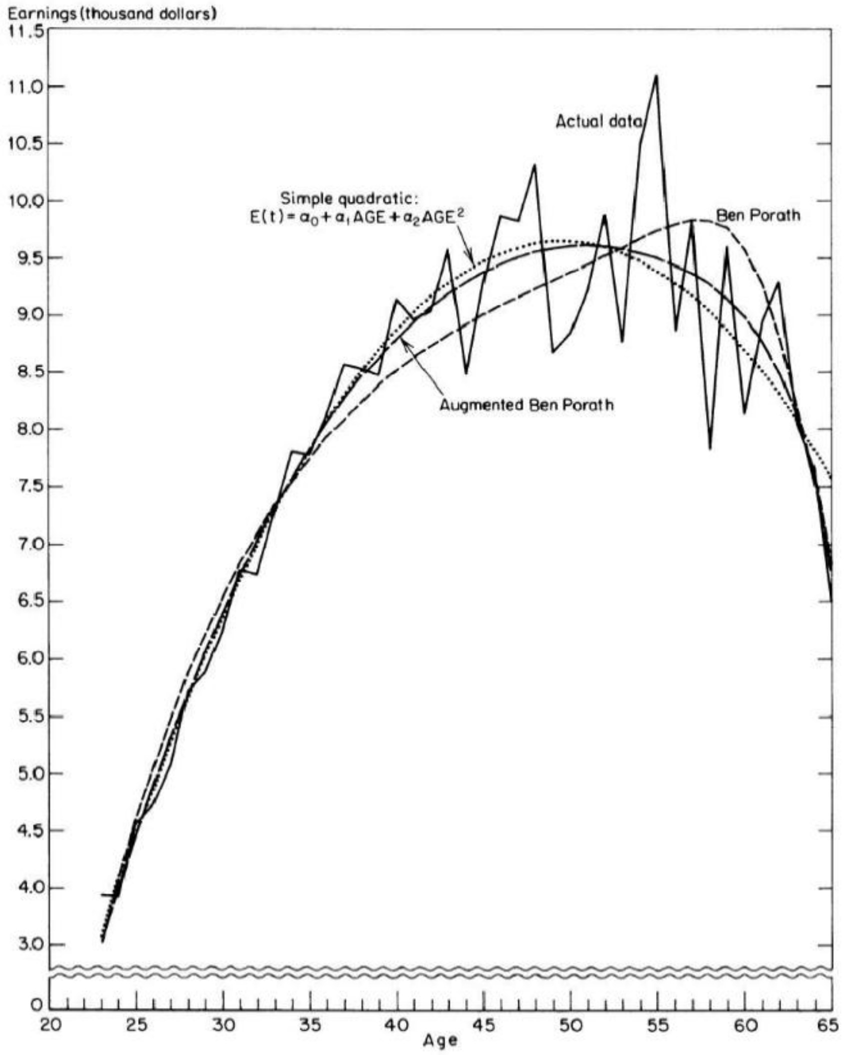


FIG. 13.—Married males, some or all college (13–16 years), not self-employed, nonfarm, not enrolled in school (1966 1/1,000 census tape).

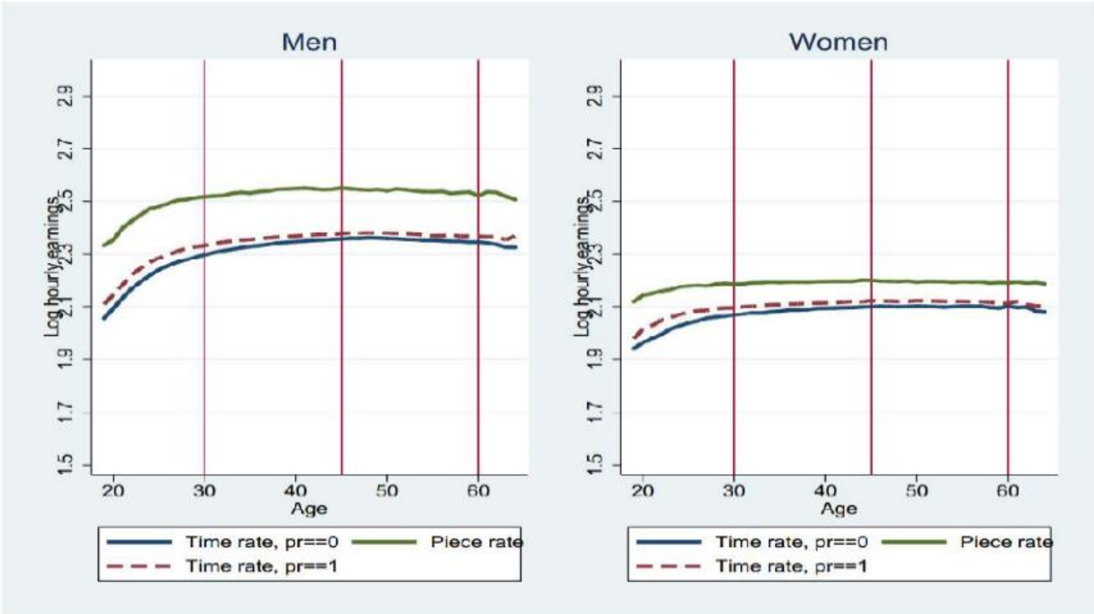


Figure 1: Age profiles of time rates and piece rates.

Pekkarinen, Tuomas and Roope Uusitalo. “Productivity-wage gaps and aging", IZA working paper (2012).

Numerical example with interior solution

In[291]:= `r = 0.05; δ = 0.03; A = 1; w = 1; α = 1 / 2; β = 1 / 8; T = 10; H0 = 20;`

$$\mathrm{A1} = \mathrm{A}\left(\frac{\alpha}{\mathrm{w}}\right)^{1-\beta}\beta^{\beta};$$

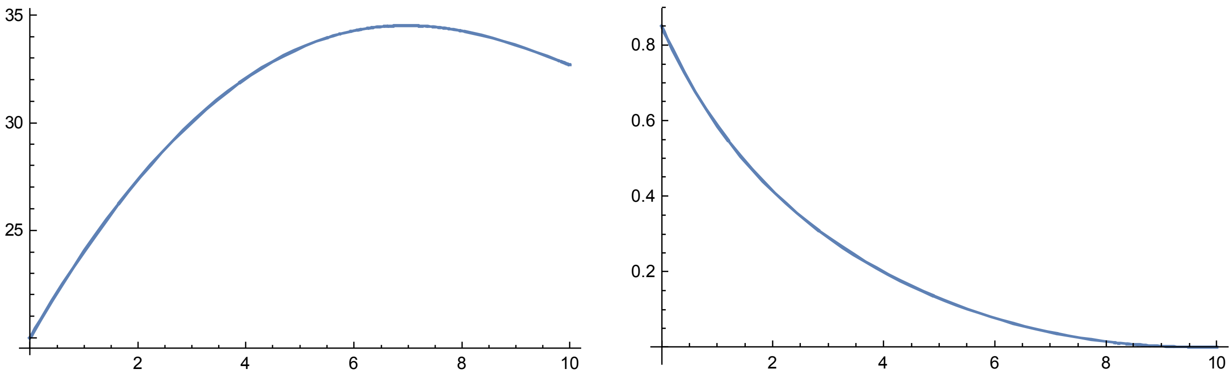
$$\mathrm{s[t\_]} = \frac{1}{\mathrm{H[t]}}\left(\mathrm{A1}\,\frac{\mathrm{w}}{\mathrm{r}+\delta}\left(1 - e^{-(\delta+\mathrm{r})\,(\mathrm{T}-\mathrm{t})}\right)\right)^{\frac{1}{1-\alpha-\beta}};$$

$$\mathrm{M[t\_]} = \frac{\mathrm{w}\,\beta}{\alpha}\,\mathrm{s[t]\;H[t]};$$

$$\mathrm{Hsol} = \mathrm{NDSolve}\left[\left\{\mathrm{H'[t]} == \mathrm{A}\,(\mathrm{s[t]\;H[t]})^{\alpha}\,\mathrm{M[t]}^{\beta} - \delta\,\mathrm{H[t]},\ \mathrm{H[0]} == \mathrm{H0}\right\},\ \mathrm{H},\ \{\mathrm{t},\ 0,\ \mathrm{T}\}\right][[1]];$$

```
GraphicsRow[Plot[#[t] /. Hsol, {t, 0, T}] & /@ {H, s}, ImageSize → Full]
```

Out[296]=

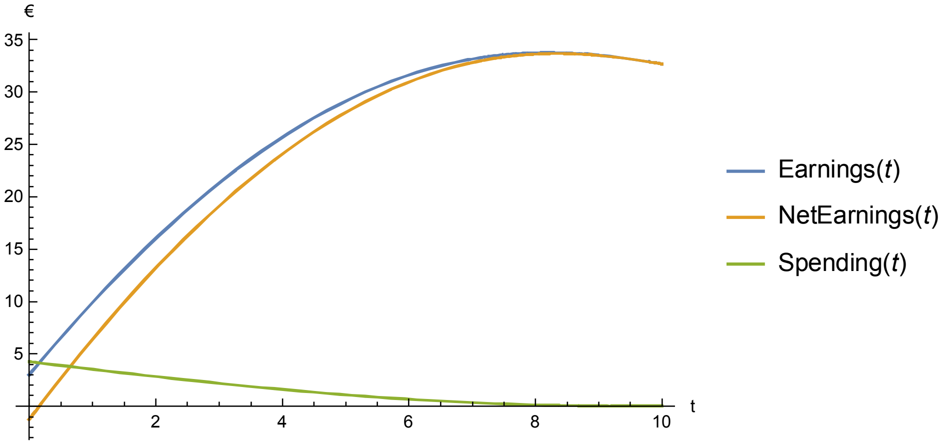


Earnings

In[297]:=
```
Earnings[t_] = w H[t] (1 - s[t]) /. Hsol;
Spending[t_] = M[t] /. Hsol;
NetEarnings[t_] = Earnings[t] - Spending[t];

Plot[{Earnings[t], NetEarnings[t], Spending[t]}, {t, 0, T}, PlotRange → All,
 PlotLegends → "Expressions", AxesLabel → {"t", "€"}]
```

Out[300]=



In[259]:=
```
NIntegrate[NetEarnings[t], {t, 0, T}]
```

Out[259]= 238.239

Note about the evolution of investment

Outside the possible specialization phase the MC of new human capital is equated with its shadow value.

The present value of a unit of *H* keeps declining as the remaining career gets shorter
MC is increasing in the quantity *Q* of human capital produced at that moment. Convex adjustment costs!

$$Q(t) = A[s(t)\,H(t)]^{\alpha}\,M(t)^{\beta}$$

$$= A[s(t)\,H(t)]^{\alpha}\left(\frac{w\,\beta}{\alpha}\,s(t)\,H(t)\right)^{\beta}$$

$$= \frac{w}{\alpha}\,A_1[s(t)\,H(t)]^{\alpha+\beta} \tag{12}$$

Investment cost *I* of using the optimal combination of inputs is

$$I = w\,s\,H + M$$

$$= w\,s\,H\left(1 + \frac{\beta}{\alpha}\right) \qquad \text{using (6)}$$

$$= w\left(Q\,\frac{\alpha}{w\,A_1}\right)^{\frac{1}{\alpha+\beta}}\left(1 + \frac{\beta}{\alpha}\right) \qquad \text{using (12)}$$

Marginal cost is

$$\frac{\partial I}{\partial Q} = w\left(1 + \frac{\beta}{\alpha}\right)\left(\frac{\alpha}{w A_1}\right)^{\frac{1}{\alpha+\beta}} \frac{Q^{\frac{1}{\alpha+\beta}-1}}{\alpha+\beta} = A_2 Q^{\frac{1-\alpha-\beta}{\alpha+\beta}}$$

where A_2 := ugly and positive.

Optimality requires e^{-rt}MC(t) = λ(t) $\Longrightarrow$

$$A_2 Q(t)^{\frac{1-\alpha-\beta}{\alpha+\beta}} = \frac{w}{r+\delta}\left(1 - e^{-(\delta+r)(T-t)}\right)$$

$$Q(t) = A_3\left(1 - e^{-(\delta+r)(T-t)}\right)^{\frac{\alpha+\beta}{1-\alpha-\beta}}$$

where A_3 := uglier and positive.

Investment declines during the working career, reaching zero at *T.* What about implied earnings?

Example with specialization phase

If the solution imples $s(0) > 1$ then $s(t) = 1$ for some $t \in [0, t^*]$.
During the specialization phase H keeps increasing the shadow value of investment equals MC

End time for specialization is a choice variable t^*

A unit of human capital at time t has the present value of earnings potential $e^{rt}\,\lambda(t)$
This exceeds the MC human capital until t^*

For this case Ben-Porath considers a CRS version with

$\beta = 0$	no "purchases"
$\alpha = 1$	CRS
$\delta = 0$	

Now MC of producing human capital is always $\frac{w}{A}$. Set it equal to $e^{rt}\,\lambda(t)$:

$$\frac{w}{r}\left(1 - e^{-r(T-t)}\right) = \frac{w}{A} \implies$$

$$t^* = T + \frac{1}{r}\log\left(1 - \frac{r}{A}\right)$$

A note about notation

JPE 1967 ⟷ This lecture note

$K_t \longleftrightarrow H(t)$

$\dot{K}_t \longleftrightarrow H'(t)$

$s_t \longleftrightarrow s(t)$

$D_t \longleftrightarrow M(t)$

$\alpha_0 \longleftrightarrow w$

$\beta_0 \longleftrightarrow A$

$\beta_1 \longleftrightarrow \alpha$

$\beta_2 \longleftrightarrow \beta$

$q \longleftrightarrow \lambda$

$\psi \longleftrightarrow \lambda e^{r t}$

Normalization

$P_d := 1$

Unchanged

r, T, Q_t, W_t, t^*

Indirect inference of earnings dynamics

Altonji, Smith, and Vidangos: “Modeling Earnings Dynamics.” (Econmetrica 2013)

What explains the age-profile of earnings? What explains cross-sectional earnings inequality?

- Key features
  - tenure and experience effects
  - transitions between jobs
  - extensive margin transitions to and from unemployment
  - intensive margin may have job-specific hours constraints
  - tenure effects
  - observed and unobserved heterogeneity of individuals and jobs
- Reduced form model motivated by theory (theories)
  - equations are “approximations” of what arise in structural models
- Main outputs: decompositions, impulse-response analysis (, proof-of-concept)

Earnings model

- Career outcomes by year: participation, hours, wages
- Exogenous variables
  - education
  - race
  - age
- Unobserved heterogeneity
  - productivity
  - switching propensity
- Latent match value variables
  - current job
  - outside job
  - unemployment
- Shock variance, persistence, measurement error, state-dependent variance, etc

See model schema on blackboard

Inference

- PSID data on male household heads 1975-1997
- Model not amenable for MLE
  - too many moving parts for a structural life-cycle model (?)
  - too complicated too jumpy (latent variables) for MLE
  - very unbalanced panel
- Method: Indirect inference

  Pick (long list of) summary statistics θ that describe the complex outcome dynamics

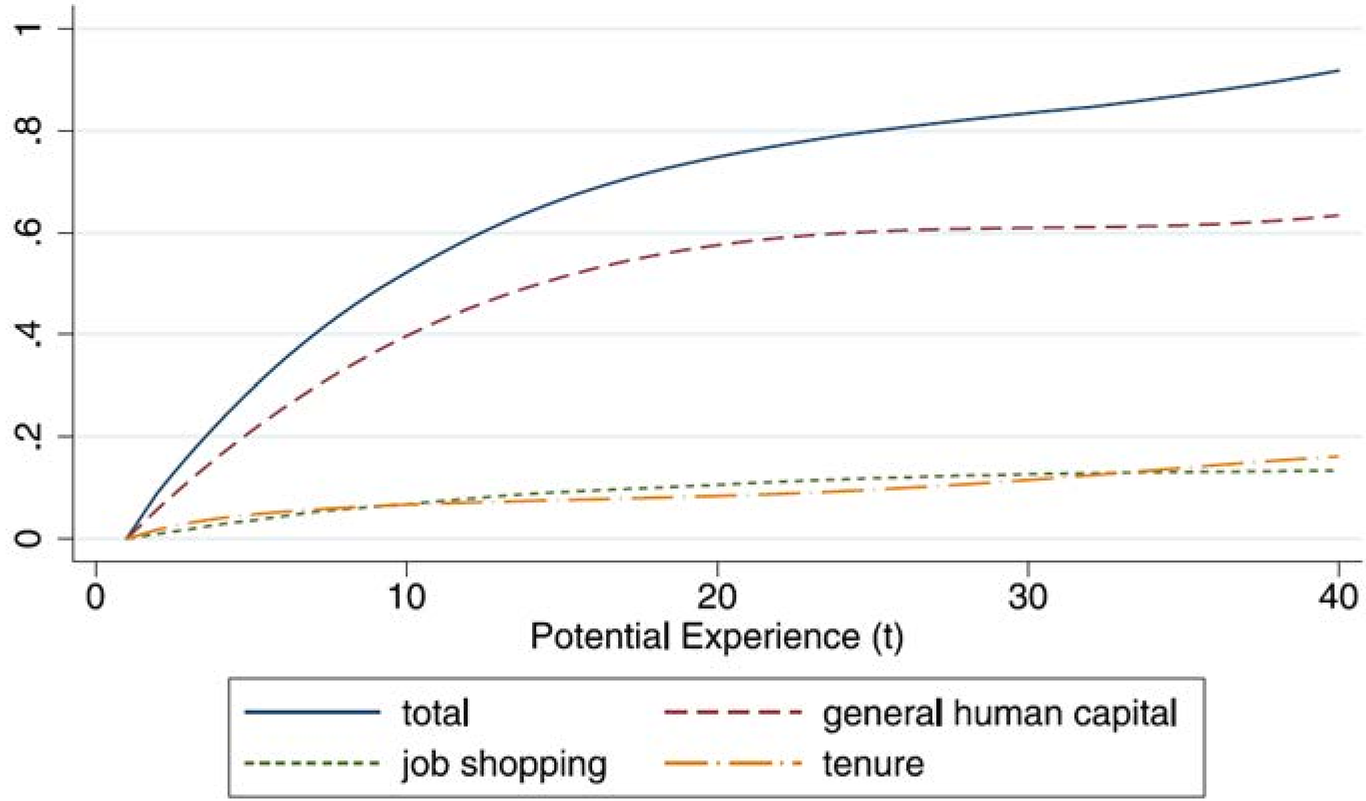
  1. Assume a value for model parameters β
  2. Simulate careers based on β and observed exogenous variables
  3. Calculate $\tilde{\theta}(\beta)$, the average θ from simulated careers

  Pick β that minimizes distance between $\tilde{\theta}(\beta)$ and the $\hat{\theta}$ calculated from actual career data

Some results /sanity checks

- Low labor supply elasticity
- Persistent wage effect of unemployment shocks decomposition (lost tenure, job-specific
- Very negative tenure effect, esp. for low-educated
- Wage offer shocks persistent across jobs
- Wage growth decomposition (job shopping, tenure, experience)
- Cross-section variance decomposition of hours, wages, earnings

Decomposition of experience profile



Log wage and potential experience. Altonji, Smith, and Vidangos (Ecma 2013)

$$\text{(1)} \qquad \text{wage}_{it} = E_{it}\text{wage}_{it}^{\text{lat}},$$

$$\text{(2)} \qquad \text{wage}_{it}^{\text{lat}} = \left[X_{it}\gamma_X^w + \gamma_{t^3}^w t^3\right] + P(\text{TEN}_{it})\gamma_{\text{TEN}}^w + \delta_\mu^w \mu_i + \omega_{it} + \upsilon_{ij(t)},$$

$$\text{(3)} \qquad \omega_{it} = \rho_\omega \omega_{i,t-1} + \gamma_{1-E_t}^\omega (1 - E_{it}) + \gamma_{1-E_{t-1}}^\omega (1 - E_{i,t-1}) + \varepsilon_{it}^\omega,$$

$$\text{(4)} \qquad \upsilon_{ij(t)} = (1 - S_{it})\upsilon_{ij(t-1)} + S_{it}\upsilon'_{ij'(t)},$$

$$\text{(5)} \qquad \upsilon'_{ij'(t)} = \rho_\upsilon \upsilon_{ij(t-1)} + \varepsilon_{ij(t)}^\upsilon.$$

(Log) wage rate determination in Altonji, Smith, and Vidangos (Ecma 2013)

Why human capital

What explains differences in earnings potential?

- HC is like physical capital because ...
  - Investment = give up something today to increase future output
  - Depreciation?
- HC is unlike physical capital because
  - Inalienability
  - Innate + acquired
  - Always lumpy