

Income Distribution and Housing Prices: An Assignment Model Approach

Niku Määttänen

Marko Terviö

ETLA and HECER*

Aalto University and HECER[†]

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Abstract

We present a framework for studying the relation between the distributions of income and house prices that is based on an assignment model where households are heterogeneous by incomes and houses by quality. Each household owns one house and wishes to live in one house; thus everyone is potentially both a buyer and a seller. In equilibrium, the distribution of prices depends on both distributions in a tractable but nontrivial manner. We show how the impact of increased income inequality on house prices depends on the shapes of the distributions, and can be inferred from data. In our empirical application we find that increased income inequality between 1998 and 2007 had a negative impact on average house prices in 6 US metropolitan areas. JEL: D31, R21.

*The Research Institute of the Finnish Economy (ETLA) and Helsinki Center for Economic Research.

[†]Aalto University School of Economics and Helsinki Center for Economic Research.

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1 Introduction

A central feature of the housing market is that housing is not a fungible commodity but comes embedded in indivisible and heterogeneous units. What people refer to as “houses,” are really bundles of land and structures. The quality of land is inherently heterogeneous because locations differ in their attractiveness due to fixed factors such as distance from the center and view of the sea. The supply of structures is more or less fixed in the short term, and only partly adjustable in the medium term. Quality of structures can also have a fixed component, due to zoning restrictions or the scarcity value of vintage architecture. At the same time, housing is a normal good, as the more wealthy spend more on housing.

Given the indivisibility and heterogeneity of houses, it is natural to consider the housing market as an assignment problem. Yet, assignment models are not a standard tool in applied work on housing economics. One reason may be that in the standard assignment model agents are ex ante divided into two distinct classes, such as “buyers” and “sellers”. The two-sidedness assumption is natural in many markets, for example when firms are matched with workers, but is problematic in the housing market where most households are potentially both sellers and buyers. Furthermore, standard assignment models assume transferable utility and so preclude income effects. In the housing market, however, income effects may be significant since housing takes up a large part of household expenditure.

In this paper, we present a one-sided assignment model with non-transferable utility, where houses are heterogeneous by quality and households are heterogeneous by income. Each household owns one house and wishes to live in one house; thus households’ reservation prices as sellers depend on the opportunities available to them as buyers. We model a single metropolitan region, where the set of households is fixed. The distributions of income and house quality are exogenous, while the distribution of house prices is endogenous, with the exception of the cheapest or “marginal” house. In general, the joint distribution of houses and income is arbitrary, which results potentially in a lot of trading between households. Equilibrium prices depend on the properties of the joint distribution, not just on the marginal distributions of income and house quality.

We show the existence of equilibrium prices under an arbitrary initial endowment. For most of the paper we focus on the equilibrium prices that emerge after all trading opportunities have been exploited. Equilibrium prices can then be solved explicitly from the joint distribution of wealth and house quality and the parameters of the utility function. We use the equilibrium conditions to derive analytical results for the impact of income distribution on the distribution of house prices.

According to our theoretical results the impact of increased income inequality on top (as well as average) house prices is ambiguous. The intuition for why increase in income inequality leads to lower prices at the bottom of the quality distribution is clear: if low-income households have less income they bid less for low-quality houses. However, in equilibrium, any changes in prices spill upwards in the quality distribution. This is because the binding outside opportunity of any (inframarginal) household is that they must want to buy their equilibrium match rather than the next best house. The equilibrium price gradient—the price difference between two "neighboring" houses in the quality distribution—is pinned down by how much the households at the relevant part of the income distribution are willing to pay for the quality difference. The price of any particular house is then given by the summation of all price gradients below, plus the price of the marginal house. Thus, while an increase in incomes at the top increases the local price gradient, the lower prices at the bottom put downward pressure on prices above. It is therefore possible for all house prices to go down in response to an increase in inequality.

Our model is readily applied to data under the assumption that all households prefer to live in their current house under current market prices. Under this "post-trade" interpretation of data the distribution of unobserved house qualities can be inferred as the distribution that gives rise to the observed price distribution as the equilibrium outcome of our model. The inferred distribution of house qualities can be used to generate counterfactuals. This setup is at its most useful in analyzing changes or policies that may be at least partly capitalized into house prices, such as income transfers and housing cost subsidies; we discuss other potential applications of our model in the end.

In the empirical application we use data from the American Housing Survey (AHS) in 1998 and 2007. We find that a suitably parametrized constant elasticity of substitution

(CES) utility function allows us to match the observed change in the price distribution under the assumption that it was caused by the observed change in incomes while unobserved house qualities remained unchanged. We consider counterfactual income distributions for 2007 where all incomes grow uniformly since 1998 at the same rate as the actual mean income in the same metropolitan region. This generates counterfactual house prices that are on average 0–10% higher, depending on the region. (Due to top coding, top 3% of the price distribution is excluded). This implies that the increase in inequality has resulted in lower prices on average than would have prevailed under uniform income growth. The contribution of uneven income growth on house prices has been positive only within the top decile, with magnitudes of up to 12%.

In the next section we discuss related literature. In Section 3 we present the model and our theoretical results. In Section 4 we show how the model can be used for inference and counterfactuals. The empirical application is presented in Section 5, and Section 6 concludes.

2 Related Literature

Assignment models are models of perfectly competitive matching markets that focus on the combined impact of indivisibilities and two-sided heterogeneity; for a review see Sattinger (1993). All other frictions, such as imperfect information or transaction costs, are assumed away. Both sides of the market are assumed to have a continuum of types, so there is no market power or bargaining as all agents have arbitrarily close competitors. Assignment models typically include an assumption of a complementarity in production, which results in assortative matching. Resulting equilibrium prices depend on the shapes of the type distributions on both sides of the market but in a reasonably tractable way. Assignment models have usually been applied to labor markets, where the productive complementarity is between job types and worker types, as in Sattinger (1979) and Teulings (1995).

There exists a large literature on two-sided assignment, where agents are ex ante divided into two distinct classes, such as "buyers" and "sellers," are matched. For recent

theoretical literature see, for example, Legros and Newman (2007) and Caplin and Leahy (2010). Our model differs from previous assignment models by featuring both one-sided matching and non-transferable utility. There is no complementarity in the usual sense, but equilibrium nevertheless involves assortative matching by wealth and house quality, essentially because housing is a normal good. We don't restrict the shapes of the distributions (beyond minimal regularity conditions). In the empirical application we can then use a nonparametric method to infer the unobserved type distribution and to construct counterfactuals similar to Terviö (2008).

In the theoretical matching literature housing has been used as the motivating example of an indivisible good that needs to be “matched” one-to-one with the buyers. The classic reference is Shapley and Scarf (1974), who present a model where houses are bartered by households who are each endowed with and each wish to consume exactly one house. They show that, regardless of the preference orderings by the households, there always exists at least one equilibrium allocation. Miyagawa (2001) extends the model by adding a second, continuous good, i.e., “money.” He shows that the core assignment of houses can be implemented with a set of fixed prices for the houses. In Miyagawa's model utility is quasilinear, so there is no potential for income effects. The results obtained in this literature are not directly applicable in our setup, as we have both indivisible and continuous goods and utility is concave.

Assignment models are rarely used in applied work on housing economics. An interesting exception is Landvoigt, Piazzesi, and Schneider (2012). They use a two-sided assignment model with concave utility to motivate a nonlinear relationship between housing quality and house prices, which they incorporate into an intertemporal model of housing demand. They employ a calibrated version of their model to study the housing market of San Diego county, California, during the 2000s boom. Their main finding is that cheaper credit for poor households increased house prices substantially at the low end of the market. Their model is a partial equilibrium one in the sense that changes in house prices do not affect the reservation prices of the sellers.

There is a long tradition in explaining heterogeneous land prices in urban economics, going back to the classic Von Thünen model, and Alonso (1964). In these models the

exogenous heterogeneity of land is due to distance from the urban center. The focus is on explaining how land use is determined in equilibrium, including phenomena such as parcel size and population density. In modern urban economics¹ there are also some models with income effects. Heterogeneity of land is modeled as a transport cost, which is a function of distance from the center, and price differences between locations are practically pinned down by the transport cost function.

More recent models with location choice and endogenous land prices include one by Epple and Sieg (1999), who estimate a structural model where public good provision is endogenous akin to Tiebout (1956). Glazer, Kanninen, and Poutvaara (2008) analyze the effects of income redistribution in a setup where heterogeneous land is owned by absentee landlords. They show that the presence of (uniformly distributed) heterogeneity mitigates the impact of tax competition between jurisdictions because taxation that drives some of the rich to emigrate also leads them to vacate high-quality land, allowing the poor to consume better land than before. In Ortalo-Magné and Prat (2010) household location choice between regions is modeled as part of a larger portfolio problem, where each region has a fixed amount of (infinitely divisible) housing capital. Different regions offer different income processes, so location decisions as well as house prices are affected by hedging considerations.

Most dynamic macroeconomic models with housing assume that housing is a homogeneous malleable good. In any given period, there is then just one unit price for housing. An exception is the property ladder structure used by Ortalo-Magné and Rady (2006) and Ríos-Rull and Sanchez-Marcos (2008), where there are two types of houses: relatively small “flats” and bigger “houses”. For our purposes, such a distribution would be far too coarse. Most macro housing literature focuses on the time series aspects of a general level of housing prices, and abstracts away from the cross-sectional complications of the market.

There is also a literature that is concerned with the dispersion of house prices between cities, while abstracting away from heterogeneity within cities. Van Nieuwerburgh and Weill (2010) study house price dispersion across US cities using a dynamic model, where

¹See, for example, the textbook by Fujita (1989).

there is matching by individual ability and regional productivity. Within each city housing is produced with a linear technology, but there is a city-specific resource constraint for the construction of new houses. This causes housing to become relatively more expensive in regions that experience increases in relative productivity. Gyourko, Mayer, and Sinai (2006) have a related model with two locations and heterogeneous preferences for living in one of two possible cities. One of the cities is assumed to be a more attractive “superstar” city in the sense that it has a binding supply constraint for land. An increase in top incomes results in more competition for scarce land, thus leading the price of houses in the superstar city to go up. Moretti (2011) presents a model where changes in relative housing prices between two cities can be affected by productivity (demand for labor) and location preference. He finds that a fifth of the observed increase in college wage premium between 1980 and 2000 was absorbed by higher cost of housing, and that the most plausible cause for this is an increase in demand for high-skill workers in regions that attracted more high-skill workers.

3 Model

We model a one-period pure exchange economy, where a unit mass of households consume two goods, housing and a composite good. Preferences are described by a utility function u , same for all households. Houses come in indivisible units of exogenous quality, and utility depends on the quality of the house, denoted by x . Every household is endowed with and wishes to consume exactly one house. A household’s endowment of the composite good y can be interpreted as its income. There are no informational imperfections, or other frictions besides the indivisibility of houses. We assume standard preferences: u is strictly increasing, differentiable, and strictly quasi-concave.

The aim is to find equilibrium prices for a continuum of house types, given the endowment and the preferences. The aggregate endowment can be described by the joint distribution of households, F_{xy} , over the consumption space, $\mathcal{S} = [x_0, x_1] \times \mathbb{R}_+$. We assume that households are distributed without gaps or atoms over $[x_0, x_1] \times [\theta_0, \theta_1]$. Figure 1 depicts the consumption space of this economy, with horizontal dimension representing the

quantile in the marginal distribution of house quality $i = F_x(x)$, and vertical dimension representing the composite good y or "consumption" for short.

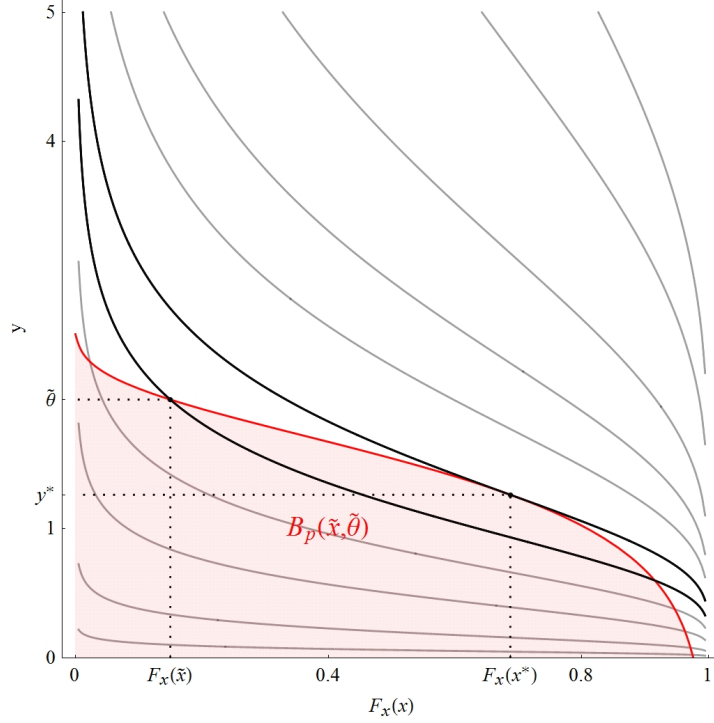


Figure 1: Consumption space. *Horizontal axes depicts the quantile of house quality $F_x(x)$ and vertical axes the level of non-housing consumption y . The shaded area is the budget set of a household with endowment $\{\tilde{x}, \tilde{\theta}\}$ and optimal choice $\{x^*, y^*\}$. A set of indifference curves is depicted (for $u(x, y) = xy$); their apparent non-convexity is due to showing the quantile of x on horizontal axes.*

Consider first the problem of an individual household. Figure 1 shows the budget curve of a household endowed with a bundle $\{\tilde{x}, \tilde{\theta}\}$, its shape determined by the price function p which households take as given. For this household, the very best houses are outside its budget set, because negative consumption is not feasible. The optimal bundle $\{x^*, y^*\}$ is where the household gets to the highest indifference curve in its budget set. This household is relatively well endowed in the composite good and would therefore give

up some of it in order to trade up to a better house. Formally, the household selects a house of type $x^* = \arg \max_x u(x, \tilde{\theta} + p(\tilde{x}) - p(x))$, thus leaving $y^* = \tilde{\theta} + p(\tilde{x}) - p(x^*)$ for consumption.

In equilibrium, the price function p must be such that *i*) all households choose their utility-maximizing x while taking p as given and *ii*) the resulting allocation is feasible. The indivisibility of houses means that the distribution of house types (the marginal distribution F_x) cannot be altered by trading. Thus feasibility requires that, for all x , the marginal density of households choosing to live in house type x is equal to the marginal density of households endowed with type x .

Equilibrium conditions To state the equilibrium conditions more formally, first define the no-trade level of income for a household endowed with a house of type x and facing price function p

$$y_p^*(x) = \left\{ y \text{ st. } x = \arg \max_{z \in [x_0, x_1]} u(z, y + p(x) - p(z)) \right\}. \quad (1)$$

We later show that, for equilibrium p , y_p^* is single-valued almost everywhere. Now denote the budget set of a household endowed with $\{\tilde{x}, \tilde{y}\}$ by

$$B_p(\tilde{x}, \tilde{y}) = \{(x, y) \in \mathcal{S} \text{ st. } p(x) + y \leq p(\tilde{x}) + \tilde{y}\}. \quad (2)$$

Finally, the demand and supply of every type of house is equal and all households are maximizing utility under prices p when

$$\int \int_{B_p(x^*, \max y_p^*(x^*))} f_{xy}(x, y) \, dx dy = F_x(x^*) \quad (3)$$

for all $x^* \in [x_0, x_1]$. (We use f to denote the density associated with F .) Then, by Walras' Law, the total consumption of y also equals its total endowment.

In terms of Figure 1, the resource constraint requires that the proportion of households with an endowment located below the budget curve for the endowment $\{x^*, y^*\}$ is equal to $F_x(x^*)$, the proportion of houses that are of quality x^* or less. The resource constraints involve double integrals, but, by discretizing the house types, the equilibrium can be solved numerically using standard methods. For most of this paper we will focus on the

post-trade allocation, which simplifies the analysis considerably, and allows us to derive closed-form results for the comparative statics of the price distribution. To get there we need two lemmas.

Lemma 1 *In equilibrium there is positive assortative matching (PAM) by household wealth and house quality.*

That is, in equilibrium, the ranking of households by wealth and by house quality must be the same. The proof is in the Appendix. In short, the diminishing marginal rate of substitution guarantees PAM: of any two households, the wealthier must live in the better house, or else the two could engage in mutually profitable trade. The twist here is that the ordering by wealth is not known beforehand, because the value of the house is endogenous. So, despite PAM, the equilibrium allocation is not obvious and depends on the shape of the joint distribution of F_{xy} . The benefit of Lemma 1 is that it guarantees that the equilibrium allocation is essentially one-dimensional, so we can index both households and houses by the house quality quantile i . We prove the existence of an equilibrium allocation in the Appendix under a finite (but arbitrarily large) number of house types. The equilibrium is associated with unique prices, up to an additive constant that can be interpreted as the opportunity cost of the worst house.

Lemma 2 *In equilibrium all households $i \in [0, 1]$ are located on a curve $\{x(i), y^*(i)\}$ in the consumption space that is continuous almost everywhere. If there are jumps they are upwards.*

This follows directly from Lemma 1: as wealth and therefore utility are increasing in i , downward jumps in y^* (as a function of i) are ruled out. Similarly, allocations supported over any thick region in consumption space would violate PAM. Only upward jumps in y^* are not ruled out, but there can only be a countable number of them or else y^* would not stay finite. Hence y^* is continuous almost everywhere.² However, y^* does not have to be increasing.³

²The jumps are where y_p^* of (1) could be multi-valued, hence the max in front of it in (3).

³For example, a social planner with a taste for equality of utility allocates less y for those who get the best x (and it is an equilibrium allocation). For details see the working paper version of this paper.

The increasing curve in Figure 2 depicts the equilibrium allocation for a lognormally distributed endowment with positive correlation. Households below the curve are the net suppliers of quality: they are endowed with a relatively high quality house and trade down in order to increase their consumption of the composite good. Households endowed with a house of quality $x(i)$ and income level $\theta(i) = y^*(i)$ do not trade. The end points of the equilibrium curve are necessarily $\{0, \theta_0\}$ and $\{1, \theta_1\}$, as the unambiguously poorest and richest households in this economy have either nothing to offer or gain in exchange.

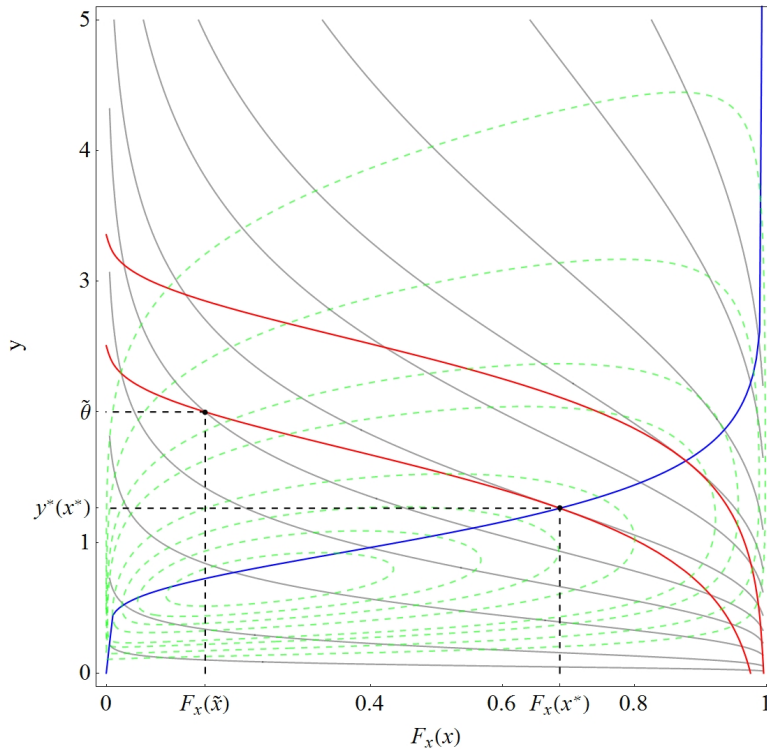


Figure 2: Equilibrium allocation. *The increasing curve shows the equilibrium allocation. Two budget curves are depicted. Dashed curves depict contours of the endowment distribution, i.e., the distribution of households before trade. This example is calculated for a lognormal distribution with $\mu_x = \mu_y = 0$, $\sigma_x = \sigma_y = 0.5$, $\rho_{xy} = 0.5$, and utility $u(x, y) = xy$.*

The resource constraints can be illustrated in terms of Figure 2. The mass of households below the budget curve that goes through $(F_x(x^*), y^*)$ is the mass of households

that, at prices p , consume houses of type x^* or worse; this is the demand for houses $x \leq x^*$. The supply of these types of houses is simply the endowment, $F_x(x^*)$, the quantile of house type x^* . An analogous condition must hold at all $x^* \in [x_0, x_1]$. Another way of illustrating the resource constraint is to note that those above the budget curve but to the left of $F_x(x^*)$ provide the net supply of houses of types below x^* , and they are trading up. This net supply of house types $[x_0, x^*]$ must equal the net demand from those trading down, which is equal to the mass of households to the right of $F_x(x^*)$ and under the budget curve.

We have now characterized what the allocation must look like after all trading opportunities have been exhausted. From now on we will restrict our analysis to this post-trade world. In a pure exchange economy, the post-trade allocation can be interpreted as just another endowment.

3.1 Equilibrium price gradient

Suppose that all trading opportunities have been exhausted, so that the current allocation is an equilibrium allocation. From now on we denote households by the quantile of their house quality, i . Let's denote by $\theta(i)$ the allocation of composite good for owners of houses of quality $x(i)$. By definition, at equilibrium prices p every household prefers to live in its own house, so that

$$i = \arg \max_{j \in [0,1]} u(x(j), \theta(i) + p(i) - p(j)) \quad (4)$$

holds for all $i \in [0, 1]$. Households are atomistic, so they take p as given. When the associated first-order condition, $u_x x' - u_y p' = 0$ is evaluated at the optimal choice $j = i$ the prices cancel out inside the utility function. (That this optimum is global is guaranteed by Lemma 1.) Solving for p' we obtain an equation for equilibrium prices:

$$p'(i) = \frac{u_x(x(i), \theta(i))}{u_y(x(i), \theta(i))} x'(i). \quad (5)$$

This price gradient is *the* key equation of our model. Combined with the exogenous boundary condition $p(0) = p_0$ it can be solved for the equilibrium price function p . The boundary condition can be interpreted as the opportunity cost for the lowest-quality

house, or as the reservation price for the poorest household stemming from some exogenous outside opportunity (such as moving to another region). The fact that x is a good makes it obvious that p is strictly increasing, and the continuity of u and x implies that p is continuous.⁴

The intuition behind the price gradient (5) is that the price difference between any neighboring houses in the quality order depends only on how much the relevant households—at that particular quantile of the wealth distribution—are willing to pay for that particular quality difference. This depends on their marginal rate of substitution between house quality and other goods, which in general depends on the level of wealth. The price level at quantile i is the sum of the outside price p_0 and the integral over all price gradients (5) below i . This is our next proposition.

Proposition 3 *Suppose θ is an equilibrium allocation. The equilibrium price function is then unique up to an additive constant p_0 and given by*

$$p(i) = p_0 + \int_0^i \frac{u_x(x(j), \theta(j))}{u_y(x(j), \theta(j))} x'(j) dj. \quad (6)$$

Note that the equilibrium price at any quantile i depends on the distributions of housing quality and income at all quantiles below i . Hence changes at any part of the price distribution spill upwards but not downwards. Loosely speaking, in terms of a discrete setup, this asymmetry in the direction of price spillovers can be understood by considering the problems faced by the richest and poorest households. If the richest household were to get even richer this would have no implication on prices, as it would not make the second richest household willing to pay more for the best house. By contrast, were the poorest household to increase its income slightly (but so that it still remained the poorest), this would increase its willingness to pay for the second worst house, thus increasing the second poorest household's opportunity cost of living in its house. This, in turn, will increase the second poorest household's willingness to pay for the third worst house, and so on, causing the local price increase at the bottom keep spilling upwards in the distribution.

⁴If θ has a discontinuity, as is allowed by Lemma 2, then p has a kink.

3.2 Comparative statics

In this section we analyze the comparative statics of equilibrium prices with respect to changes in income distribution. Here we assume that the economy begins in an equilibrium where there is a strictly positive relation between income θ and house quality. For brevity, we call this a "regular" equilibrium.

Definition. A *regular equilibrium allocation* is one where there is a strictly monotonic increasing relation between household income and house price.

This, in our model, is equivalent to the case where income and wealth are perfectly rank correlated. Due to PAM (Lemma 1) the equilibrium in our setup has to satisfy perfect rank correlation between wealth and house price, which is a weaker requirement.⁵ The purpose of this simplification is to make sure that the analytics of the equilibrium allocation can be used even under changes in income distribution, as order-preserving changes in incomes are then guaranteed to keep the ranking of households by wealth unchanged. A change in the ordering by wealth would generate trading and would thus not fall within the scope of the no-trade case.

It is worth noting that an increase in income inequality does not necessarily lead to an increase in the prices of best houses.

Proposition 4 *Suppose that the endowments form a regular equilibrium allocation, and that the income distribution experiences a mean-preserving and order-preserving spread where incomes decrease below quantile $h \in (0, 1)$ and increase above h . Then housing prices will either i) decrease everywhere or ii) decrease everywhere except at quantiles $(h', 1]$, where $h' > h$.*

Proof. Denote the new distributions by hats. By definition, the new income distribution

⁵In our empirical application the monotonic relation between income and house prices emerges very naturally as a "side-effect" of kernel smoothing the data under the minimal assumptions needed to make wealth monotonic in house price.

satisfies

$$\begin{aligned}\hat{\theta}(i) &< \theta(i) && \text{for } i \in [0, h), \\ \hat{\theta}(i) &> \theta(i) && \text{for } i \in (h, 1], \\ \int_0^1 (\hat{\theta}(j) - \theta(j)) \, dj &= 0.\end{aligned}$$

Applying (6), the change in prices at any $i \in [0, 1]$ is

$$\begin{aligned}\hat{p}(i) - p(i) &= p_0 + \int_0^i \frac{u_x(x(j), \hat{\theta}(j))}{u_y(x(j), \hat{\theta}(j))} x'(j) \, dj - \left(p_0 + \int_0^i \frac{u_x(x(j), \theta(j))}{u_y(x(j), \theta(j))} x'(j) \, dj \right) \\ &= \int_0^i \left(\frac{u_x(x(j), \hat{\theta}(j))}{u_y(x(j), \hat{\theta}(j))} - \frac{u_x(x(j), \theta(j))}{u_y(x(j), \theta(j))} \right) x'(j) \, dj.\end{aligned}\tag{7}$$

The inverse of the marginal rate of substitution between y and x , $u_x(x, \theta)/u_y(x, \theta)$, is increasing in θ , and $x' > 0$, so the integrand in (7) is negative at all j where $\hat{\theta}(j) < \theta(j)$, i.e., for $j < h$. Similarly, the integrand is positive for $j > h$. The definite integral in (7) must therefore be strictly negative at $i = h$, where it reaches its minimum, and increasing above h . If, at some $h' > h$ the definite integral reaches zero then it will be positive at all $i > h'$, but it might not reach zero before $i = 1$, in which case the price change is negative at all $i \in (0, 1]$. ■

Intuitively, if income is redistributed from poor to rich, this will increase the local price gradient (5) at the top quantiles, as the willingness-to-pay for extra quality goes up for the rich. But, for the same reason, the price gradient at bottom quantiles goes down. Above h , the change in the price gradient is positive, but the negative spillover from below will dominate until some $h' > h$, and it can be that the cumulative impact of positive gradients is not enough to overtake the negative impact. It is therefore possible for all house prices to go down in response to an increase in inequality. This would happen, for instance, if the quality differences between best houses are relatively small so that the rich are less inclined to use income growth to bid up the price of the next better house.

The impact of a simple increase in income levels is characterized by the next Proposition.

Proposition 5 *Suppose the endowments form a regular equilibrium allocation. If all incomes rise in an order-preserving manner, then housing prices will increase at all quantiles $i \in (0, 1]$ and this increase is increasing in i .*

The reasoning is similar as in the Proof of Proposition 4, but even simpler because the new price gradient is greater than the original gradient at all quantiles.

As an immediate corollary, an increase in income levels will also increase the variance of house prices. A simple extreme case is where all incomes catch up with the highest income level. Proposition 4 tells us that such a complete elimination of income inequality would increase both the levels and the dispersion of house prices. Intuitively, all households would then be competing for the best houses and the price difference between a low quality and a high quality house must become relatively large to make some household willing to hold the low quality house.

The exogenous bottom price $p(0)$ is held constant under comparative statics, so the price changes we are referring to are for quantiles $(0, 1]$. A change in the lowest price, p_0 , would cause all prices to change by that same amount. As the model does not include the possibility to move out of the market, the level of the constant is irrelevant for the attractiveness of any trade: it increases both the buying and selling prices, and thus washes out of all transactions.

3.3 Absentee landlords

In urban economics models with heterogeneous land it is standard to assume that all land is initially owned by competitive outside sellers or “absentee landlords.” This is similar in spirit to two-sided matching in that, by construction, sellers’ reservation prices can be considered exogenous to the problem. The absentee landlord assumption can be introduced to our model by assuming that the revenue from house sales goes to atomistic outside agents who are not buyers in this market and have no market power as sellers.

Consider the household at quantile i of the income distribution. Again, by Lemma 1, equilibrium must involve positive assortative matching by wealth, which now consists only of income θ , and house quality. Thus p must result in every household buying a house of the same quality rank as is their rank in the income distribution.

$$i = \arg \max_{j \in [0,1]} u(x(j), \theta(i) - p(j)) \quad \text{for all } i \in [0, 1] \quad (8)$$

Here the price of the house actually chosen is not part of household wealth and so it does

not cancel out of the price gradient. As a result, equilibrium prices are now defined as a nonlinear ordinary differential equation:

$$p'(i) = \frac{u_x(x(i), \theta(i) - p(i))}{u_y(x(i), \theta(i) - p(i))} x'(i). \quad (9)$$

Combined with a boundary condition this can still be solved for the equilibrium price function p . Now the boundary condition must satisfy $p_0 \leq \theta(0)$, or else the poorest household cannot afford to live anywhere.

If all houses were owned by an absentee monopolist, then the self-selection constraint (9) would still have to hold, but the seller could restrict the quantity sold. This would require the monopolist to be able to credibly commit to not selling the lowest quality houses up until some quantile $m \in (0, 1)$. The lowest price $p(m)$ would then have to be pinned down by some outside opportunity for the buyers (e.g. an exogenous utility level from living in another region).

3.4 The case with CES

For the empirical application we assume CES utility,

$$u(x, y) = (\alpha x^\rho + (1 - \alpha) y^\rho)^{\frac{1}{\rho}}, \quad \text{where } \rho < 1 \text{ and } \alpha \in (0, 1), \quad (10)$$

with Cobb-Douglas utility defined in the usual fashion at $\rho = 0$. If wealth is just equal to income θ (as in the absentee landlord setup above), then the equilibrium price gradient (9) is

$$p'(i) = \frac{\alpha}{1 - \alpha} \left(\frac{\theta(i) - p(i)}{x(i)} \right)^{1-\rho} x'(i). \quad (11)$$

Under the post-trade assumption, where wealth is equal to the sum of income and the equilibrium value of one's house, $\theta(i) + p(i)$, the prices cancel out in the right-hand side and this can be solved as

$$p(i) = p_0 + \frac{\alpha}{1 - \alpha} \int_0^i \left(\frac{\theta(s)}{x(s)} \right)^{1-\rho} x'(s) ds. \quad (12)$$

When p and θ are observed, then x can be solved for under a given elasticity parameter ρ . The other preference parameter, α , is absorbed by the units of x and can then be normalized at, say, one half.

4 Inferring quality and computing counterfactuals

For the equilibrium price relation to be applied to cross-sectional data it is necessary to assume that observed prices correspond equilibrium prices that emerge after all trading opportunities have been exhausted. We think this is a reasonable interpretation of data because, at any point in time, only a small fraction of households trade houses. Under this assumption the unobserved distribution of house qualities can be inferred from the observed relation between household income and house prices. That is, for given distributions p and θ , and for a given utility function u , the distribution of x can be inferred. (We discuss the inference of preferences in the next section.) This can be done by treating x as the unknown in the differential equation (5), while normalizing the boundary condition $x(0) = x_0$ at any positive value.

Having inferred the distribution of x based on the observed distributions of θ and p , any counterfactual income distribution $\hat{\theta}$ then pins down a counterfactual distribution of house prices, by method of combining $\hat{\theta}$ and x in the (possibly discrete equivalent of) equilibrium price relation (12). Note, however, that as p_0 is exogenous, our model only explains the differences in prices relative to the marginal unit of housing, $p - p_0$. In the counterfactuals the lowest price is always taken to be the lowest price in the data.

Inference requires data on prices (house values) $p_0 < p_1 < \dots < p_N$ and associated household incomes θ_h . The minimal requirement for the data to be consistent with the model is that observed wealth $p + \theta$ must be in strictly positive relation with p (Lemma 1). This relation is, of course, not perfect in reality, but it emerges very naturally in our data with kernel smoothing (details of which are explained in the Appendix).

The basis for inference is the "incentive compatibility" condition that makes household h want to buy its equilibrium match—which is house h —instead of any other house h' . For CES utility, these conditions are

$$(\alpha x_h^\rho + (1 - \alpha) (\theta_h)^\rho)^{\frac{1}{\rho}} \geq (\alpha x_{h'}^\rho + (1 - \alpha) (\theta_h + p_h - p_{h'})^\rho)^{\frac{1}{\rho}}, \text{ for all } h, h' \in \{0, \dots, N\}. \quad (13)$$

Thanks to PAM, we know that this constraint is only binding for household pairs that are "neighbors" in the rank by house quality ($h' = h - 1$). For convenience, we assume

that these constraints hold as an equality, and we obtain a discrete equivalent of the price gradient.⁶

$$p_h = p_{h-1} + \left((\theta_h)^\rho + \frac{\alpha}{1-\alpha} (x_h^\rho - x_{h-1}^\rho) \right)^{\frac{1}{\rho}} - \theta_h \quad (14)$$

Equivalently, solving for x_h yields the inference formula

$$x_h = \left(x_{h-1}^\rho + \frac{1-\alpha}{\alpha} [(\theta_h + p_h - p_{h-1})^\rho - (\theta_h)^\rho] \right)^{\frac{1}{\rho}}. \quad (15)$$

Note that the value of $\alpha \in (0, 1)$ is without consequence when x is unobserved, because changing α is equivalent to changing the units of x . We set $\alpha = 1/2$. (The levels of x are not themselves meaningful, and this normalization is without loss of generality for the counterfactual prices). Denoting $\hat{x} = x^\rho$, (15) can now be solved as

$$\hat{x}_h = \hat{x}_0 + \sum_{j=1}^h [(\theta_j + p_j - p_{j-1})^\rho - (\theta_j)^\rho], \quad (16)$$

which includes an undefined constant of integration \hat{x}_0 , i.e. the quality of the worst occupied house.

Finally, by combining inferred \hat{x}_h with observed p_0 , and posited counterfactual $\hat{\theta}_h$ in the price difference equation (14), we obtain counterfactual prices \hat{p}_h . In the CES case the steps can be combined to yield the counterfactual prices directly as

$$\hat{p}_h = \hat{p}_{h-1} + \left((\theta_h + p_h - p_{h-1})^\rho - (\theta_h)^\rho + (\hat{\theta}_h)^\rho \right)^{\frac{1}{\rho}} - \hat{\theta}_h \quad (17)$$

for $h = 1, 2, \dots$, with $\hat{p}_0 = p_0$ again taken as exogenous.

5 Empirical application: income inequality and house prices

5.1 Data

We use income and price data from the American Housing Survey (AHS) for six metropolitan areas (MA, or "city"): Baltimore, Boston, Houston, Minneapolis, Tampa, and Wash-

⁶In the discrete setup there is a match-specific rent that the "neighbors" could bargain over. Here we in effect assume that "sellers" have all bargaining power. Making the opposite assumption makes practically no difference to our empirical results.

ington. Our income measure is total disposable income, including taxes and transfers, during the last year. "Houses" include homes in multi-unit dwellings. The choice of MAs was determined by the availability of data. For a more detailed description of the data, see Appendix B.

In the AHS data, the house prices have been censored separately at each MA at the 97% percentile. Thus we have to exclude the top 3% from our analysis. Furthermore, there are apparently significant data quality issues at the bottom of the price distributions, with many house prices observed in the range of a few hundred or thousands of dollars. For this reason we drop the bottom 5% of houses in each MA, and so the 5th percentile price will be the lowest (and thus exogenous) house price in our analysis. All of our results refer to this restricted sample.

We observe the joint distribution of income θ and house price p in each of the six MAs for both 1998 and 2007. There is a strong positive relationship between house value and income. Rank correlation between income and house price is between 0.38 and 0.52 depending on MA (see Table B1). To be consistent with the equilibrium of our model, the levels of wealth $w = \theta + p$ should be perfectly rank correlated with house value p (recall Lemma 1). Thus we reduce the relation of income and house price into a curve, by using kernel regression to estimate $\theta(i)$ as $E[\theta|F_p(p) = i]$, where F_p is the empirical CDF of p .⁷

Figure 3 displays the distributions of smoothed income relative to its mean in 1998 and 2007. The only apparently exceptional case is Houston where lowest incomes have grown faster than the mean. Table B1 displays Gini coefficients for our estimated permanent income (i.e. smoothed income), annual income, and house value. All MAs feature an increase in income inequality from 1998 to 2007. Housing values show no such systematic pattern.

We need to set a value for the interest rate r to make the units of yearly income compatible with the house price: income θ is measured as annual income divided by the interest rate. (In effect, we assume that households face an infinite time-horizon, and expect no changes in the future.) We set $r = 0.05$; changing r over a reasonable range

⁷We use the Epanechnikov kernel and a bandwidth of 9%, except in Tampa where a bandwidth of 11% is required for the smoothed data to conform to assortative matching by wealth and house price.

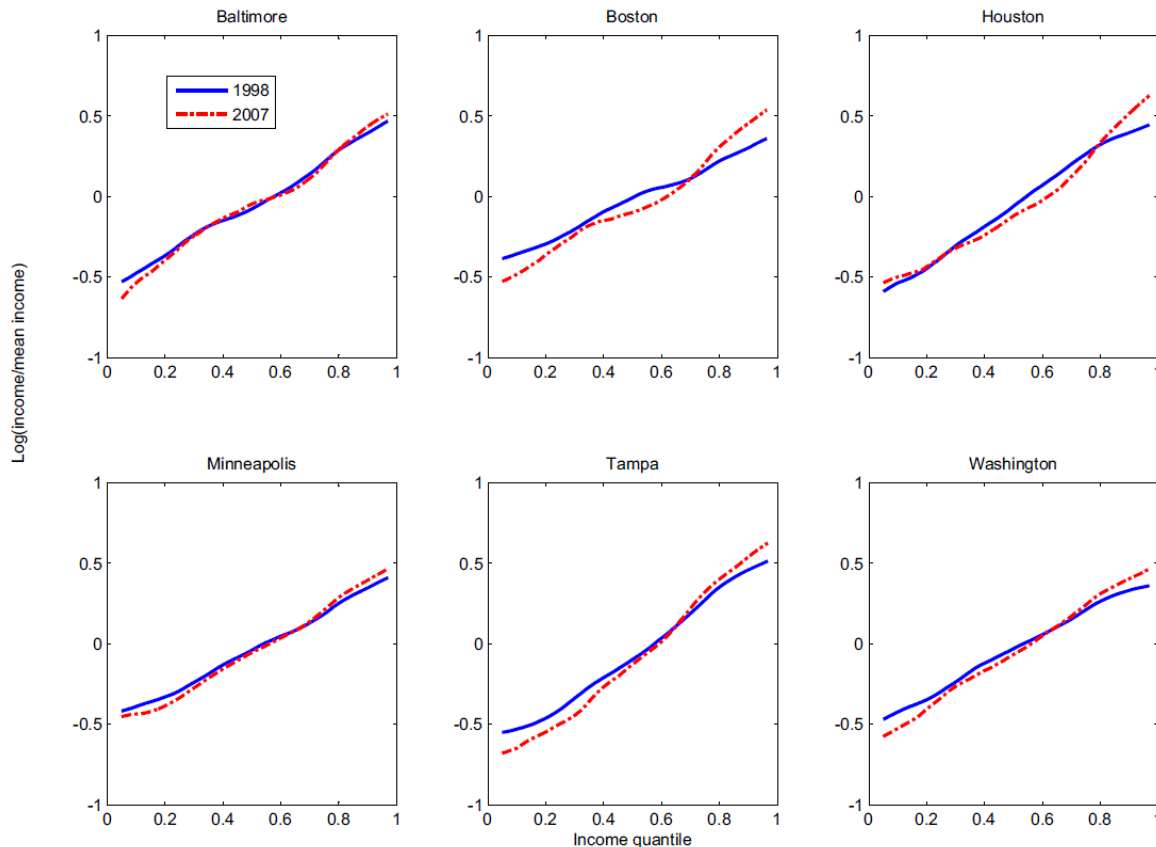


Figure 3: *Distribution of permanent income relative to mean by metropolitan area.*

(2 – 8%) makes little difference to our inferred elasticity parameter ρ or the results of the counterfactual experiments that we present below.

5.2 Inferring preferences

To apply the model we need a value for the preference parameter ρ in the CES-utility function (10).⁸ We choose ρ using the following procedure. First, for a given ρ and for the observed $\{\theta^{98}, p^{98}\}$, we infer the quality distribution in 1998, denoted by x^{98} , using the inference formula in (16). Then we then use the equilibrium price formula—the dis-

⁸Li, Liu and Yau (2009) and Bajari, Chan, Krueger and Miller (2010) use a life cycle model to estimate the same preference parameter. However, as far as we know, we are the first to exploit changes in the cross-sectional distribution of housing prices to estimate household preference parameters. This is possible in our model because housing prices are in general a non-linear function of housing quality.

cretized equivalent of (12)—to predict the 2007 housing price distribution given x^{98} and the observed wealth in 2007 ($\theta^{07} + p^{07}$). In other words, assuming that the quality distribution did not change over time, we ask what would be the predicted price distribution in 2007 given the observed 2007 income distribution. We denote this prediction by \hat{p}^{07} . We have to implicitly assume that the ranking of households by income has not changed, as we apply a static model in both years. We repeat this for a range of values for ρ , and pick the value that gives the best prediction for the 2007 price distribution.⁹

In the benchmark case we infer ρ separately for each MA, but we also report the case where ρ is restricted to be common for all MAs. Table B2 shows for each MA the inferred value for the elasticity of substitution. Notice that we are trying to match the entire 2007 price distribution with just two parameters, the other being the bottom price p_0 (5th percentile in the full sample) which we match exactly by construction.

Figure 4 shows the relative prediction error, $\log(\hat{p}^{07}/p^{07})$, by quantile of house value. The model seems to provide a reasonable fit in most cities. The exceptions are Tampa and Washington. There, according to the model, prices of the best houses should have increased much more than they actually did. It may be that the supply of top quality housing is relatively elastic in these two cities, whereas we assume a fixed quality distribution. At least for Tampa, it seems plausible that there are relatively attractive undeveloped locations, while in a city like Boston the best locations are mostly already either built up or protected from development. Elastic housing supply is also an example of an omitted factor that would bias our estimate of the elasticity parameter. However, as we show below, the results of our counterfactual experiment are relatively insensitive to the value of the elasticity parameter.

5.3 Counterfactuals

We construct counterfactuals to assess the impact of changes in the income distribution between 1998 and 2007 on housing prices in six US metropolitan areas. Specifically,

⁹We minimize the mean of absolute percentage errors (MAPE), i.e. the mean value of $|\log(p^{07}(i)) - \log(\hat{p}^{07}(i|\rho))|$. In the common ρ case we use the unweighted average of MAPE across the MAs.

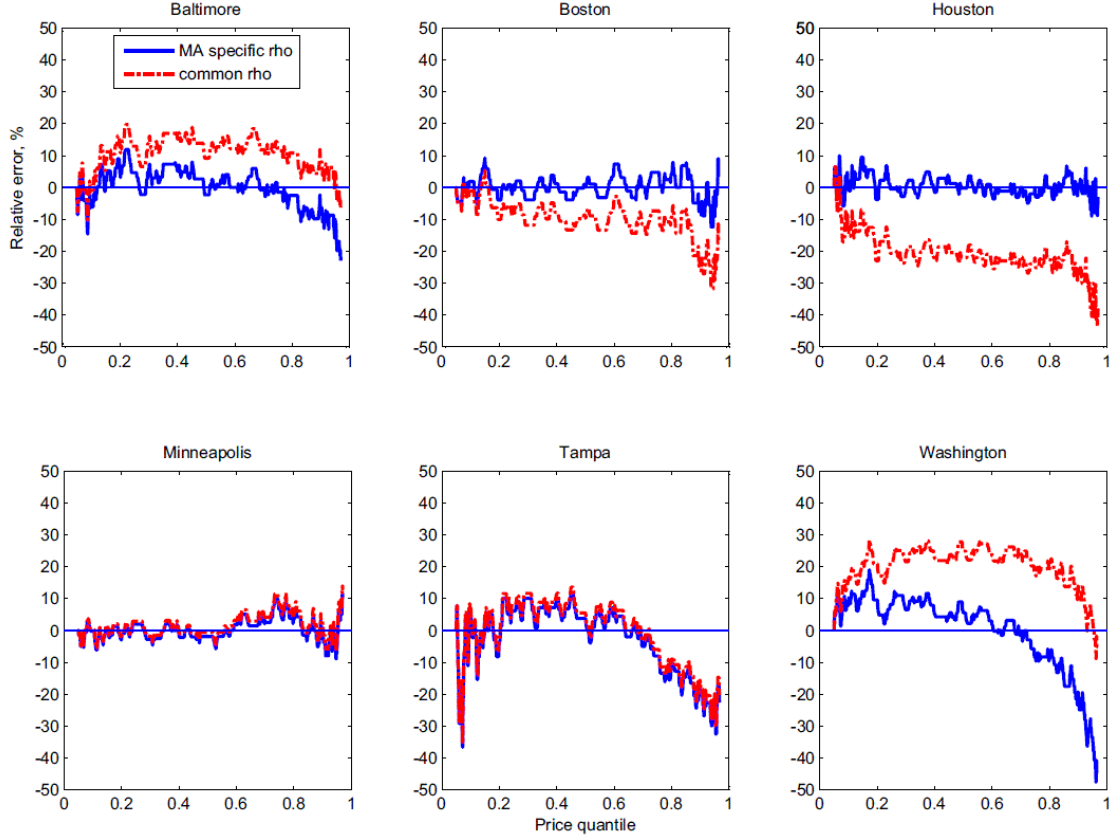


Figure 4: *Errors in predicted 2007 house prices by metropolitan area. Solid blue line refers to the baseline case, where preferences are estimated separately in each MA, and dashed red line to the case where preferences are restricted to be the same everywhere.*

given the inferred values for ρ (see Table 2) and the quality distributions x^{98} in each MA, we compute the predicted price distributions in 2007 under a counterfactual income distribution that has the same shape as in 1998 but the same mean as in 2007. We then compare this counterfactual price distribution with the fitted price distribution that is obtained by plugging in the actual 2007 income distribution; the difference between the two is our estimate of the contribution of impact of increased inequality on house prices.

Figure 5 displays the relative difference between counterfactual and empirical housing prices. Consider first the benchmark case. The impact of increased income inequality is qualitatively similar in all but one MA: it has lowered housing prices until about 80-90th percentile and increased prices in the upper tail. The exception is Houston, where

changes near the bottom of the income distribution work to increase the prices of the lowest quality houses. This reflects the fact, visible in Figure 3, that in Houston, unlike in the other MAs, the lowest incomes have increased relative to the mean income, even though the Gini coefficient has increased there as in other MAs.

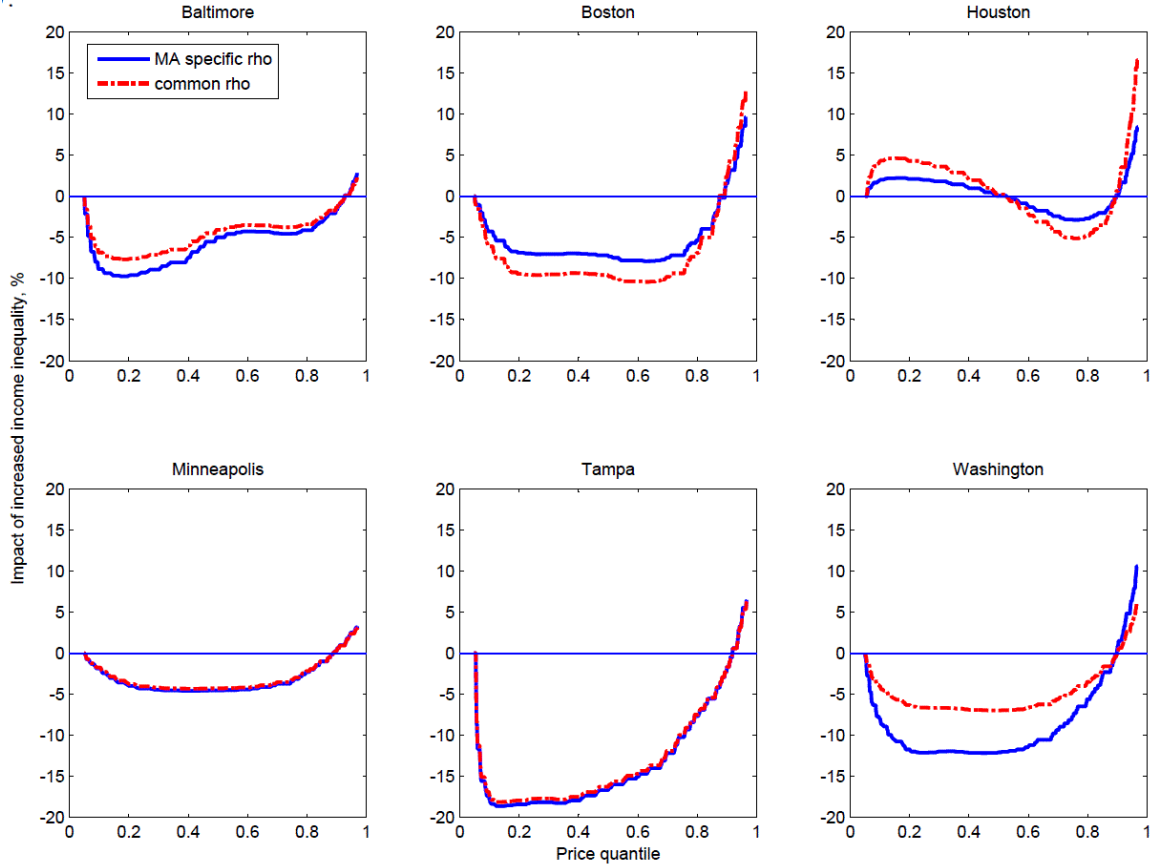


Figure 5: *The impact of increased income inequality on housing prices.*

By and large, the results are similar in the case where the elasticity parameter is the same for all MAs. In particular, the point at which the price effect turns from negative to positive is almost the same in both specifications. Also the increase in the price of even the best houses (those at the 97th percentile) is always moderate. Intuitively, the increase in income inequality results in lower incomes at the bottom of the distribution relative to the counterfactual of uniform income growth. This works to lower the prices of the lowest quality houses. As explained in Section 3.2, this negative price impact at the bottom spills upwards in the quality distribution. This effect counteracts the local

increase in willingness-to-pay among the lower part of the better-off households who in actuality saw their incomes rise faster than the mean.

The impact on average prices is shown in Table 1. The first two columns display the mean relative effect of the change in the shape of the income distribution on house prices for the MA-specific and common preference specifications. The third column displays the absolute price change in the benchmark case. The mean impact of change in income inequality in the benchmark specification varies from -10.1% in Tampa to 0.2% in Houston.¹⁰

	Change (%)		Change (\$k)
	benchmark	common ρ	benchmark
Baltimore	-4.4	-3.6	-16.0
Boston	-4.5	-5.9	-23.3
Houston	+0.2	+0.6	+2.7
Minneapolis	-2.6	-2.5	-7.7
Tampa	-10.1	-9.9	-27.6
Washington	-6.6	-4.0	-36.7

Table 1. Change: The impact of the change in the shape of the income distribution on house prices between 2007 and 1998, relative to what house prices would have been under uniform income growth.

When interpreting these results it is important to keep in mind that we keep all other things, including the distribution of house quality, fixed. In reality the value of the house consists of the value of land and the value of structures and it is, of course, possible to adjust the quality of the structures to some extent. We abstracted away from adjustable quality and concentrated on relatively short term variation in our empirical application. However, even in the long term, according to Davis and Heathcote (2007), most growth and volatility in U.S. house prices has been due to variation in land values.¹¹

¹⁰In a working paper version (Määttänen and Terviö 2009) we used data from the Helsinki metro region in 1998 and 2004, where there was also a significant increase in income inequality. The results are similar, with a mean effect of -3.2% .

¹¹Davis and Palumbo (2008) estimate the share of land in house values for detached single-family

6 Conclusion

We have presented a new framework for studying the relationship between the income distribution and the housing price distribution. In the model houses are heterogeneous and indivisible. The key element is that all houses are owned by the households, and, due to concave utility, their reservation prices as sellers depend on the opportunities available to them as buyers, which in turn depend on their incomes and on the prices of other houses. Thus, our model provides a framework for analyzing how income differences get capitalized into house prices. The equilibrium is tractable under the assumption that households prefer to live in their current house.

The equilibrium can be understood intuitively by considering the price gradient, which is, loosely, the price difference between “neighboring” houses in the quality distribution. The price gradient expresses how much households that inhabit a particular part of the quality distribution in equilibrium are willing to pay for the quality difference over next best house. This depends on their marginal rate of substitution between house quality and other goods, which in general depends on their level of wealth. The price level at any quantile in the distribution is the sum over all price gradients below.

The natural comparative static in our setup is order-preserving changes in income distribution, as these have an impact on prices without generating trading. The model yields a number of theoretical implications about the relation of income and house price distributions. Any increase in income levels will increase both the level and dispersion of house prices, but an increase in income inequality will decrease house prices except (possibly) in a segment adjacent to the top. House prices at the top can go either way in response to an increase in income inequality, depending on the details of both supply and demand sides of the market.

As an illustration of the model, we obtained a theory-driven estimate for the impact of recent increases in income inequality on house prices in the US. The equilibrium conditions of the model allow us to estimate the housing quality distribution without imposing restrictions on the shapes of the distributions. Specifically, we asked how the 2007 price

homes; in our sample cities it ranges between 31% (Houston) and 76% (Boston) in 2004.

distribution would differ from the actual price distribution if income of every household would have grown at the actual mean rate since 1998. We found that the impact of increased inequality on prices has been modest but negative on average, and positive only at the top decile. This is because the cumulative impact of reductions in the price gradient at the bottom households, whose income growth did not keep up with the mean growth, dominates the positive effects almost all the way to the top of the distribution.

Our model opens the possibility for other applications. Most directly, it is a natural framework for studying how various housing and income subsidy schemes impact housing prices. The intuition of the price gradient makes it clear why housing subsidies targeted for the poor will not merely be capitalized into prices of low-quality housing but will spill upwards in the quality ladder. A serious empirical analysis of this issue will require the inclusion of non-owner-occupied housing. Other important and challenging directions include migration (choice between cities) and preference heterogeneity.

The counterfactuals presented in this paper took the form of order-preserving changes in the exogenous income distribution. This was crucial for being able to apply the formulae derived under the assumption of no-trade equilibrium. The lack of trading also meant that price changes do not have welfare effects—they are merely changes in paper wealth. In order to have welfare effects, price changes have to generate trading. One interesting and challenging topic for further study is the welfare effects of regulations, for instance transaction taxes, which can be expected to distort the matching of houses and households. A particularly policy-relevant issue is the impact of repealing rent control, which would result in a simultaneous supply and demand shock. Again, the impact on the entire distribution of housing prices is likely to be nontrivial. Our characterization of the (no-trade) equilibrium should be helpful even in applications that involve trading, although these will require more involved numerical methods.

Appendix A: Proofs

Proof of Lemma 1: Positive assortative matching by wealth and house quality

Define the reservation price $\Delta(x'|x, y)$ by

$$u(x, y) = u(x', y - \Delta), \quad (18)$$

so that Δ is the maximum price that a household with endowment $(x, y) \in \mathbb{R}_+^2$ is willing to pay to switch into a house of type x' . Thus $\{x', y + \Delta(x'|x, y)\}$ traces an indifference curve that goes through the endowment (x, y) ; it is strictly decreasing, and Δ changes sign at $x' = x$. The reservation price of switching into a worse house is negative.

Consider two households, $h = 1, 2$, endowed with houses $x_1 < x_2$. There is trade between them if and only if their reservation prices for the trade sum up to something positive, i.e., if

$$\Delta(x_2|x_1, y_1) + \Delta(x_1|x_2, y_2) > 0. \quad (19)$$

We want to show that if $x_1 < x_2$ and $u(x_1, y_1) > u(x_2, y_2)$ then there has to be trade, which this rules out any violations of PAM in equilibrium. The wealthier household has a larger budget set, so PAM by wealth and house quality is equivalent with PAM by utility and house quality.

Consider a point $\{x_2, y'_2\}$ where $y'_2 = y_1 - \Delta(x_2|x_1, y_1)$. This is along the indifference curve going through (x_1, y_1) and vertical in relation to (x_2, y_2) . Swapping positions between (x_2, y'_2) and (x_1, y_1) does not change utility so if $y_2 = y'_2$ then $\Delta(x_1|x_2, y'_2) + \Delta(x_2|x_1, y_1) = 0$. If $\Delta(x_1|x_2, y)$ is strictly decreasing in y then there will be trade if and only if $y_2 < y'_2$, which is equivalent to $u(x_1, y_1) > u(x_2, y_2)$. Differentiating (18) we obtain

$$\frac{d\Delta}{dy} = 1 - \frac{u_y(x_2, y)}{u_y(x_1, y - \Delta)}. \quad (20)$$

We know that u_y is positive and decreasing for a fixed x , but now the comparison is at two different levels of x , yet on the same indifference curve. Use $\tilde{y}(x)$ to denote the indifference curve. Thus (20) is negative if

$$\frac{d}{dx} (u_y(x, \tilde{y}(x))) = u_{xy} + u_{yy} \left(\frac{d\tilde{y}(x)}{dx} \right) = u_{xy} - u_{yy} \left(\frac{u_x}{u_y} \right) \geq 0 \quad (21)$$

This is just the condition for the marginal rate of substitution MRS_{yx} to be decreasing in y (i.e., for MRS_{xy} to be increasing in y), which holds for any quasi-concave utility function.

Proof: Existence of equilibrium Let's now discretize the house types, so that $0 < x_0 < \dots < x_N < \infty$. For brevity, we will refer to households endowed with a type- k house as households of type k . Denote the mass of type- k households with income equal or lower than θ by $F_k(\theta)$. We assume that all of these conditional income distributions are continuous with full support $[\underline{\theta}, \bar{\theta}]$, where $0 < \underline{\theta} < \bar{\theta} < \infty$. The mass of type- k households is $m_k > 0$, so $F_k(\bar{\theta}) = m_k$. Recall that we have normalized the total mass at $\sum m_k = 1$.

With discretized house types we require an additional assumption:

$$u(x_{k-1}, \bar{\theta}) > u(x_k, \underline{\theta}) \text{ for all } k = 1, \dots, N. \quad (22)$$

This means that distribution of autarky utility has overlap between neighboring household types. Together with Lemma 1, this implies that there must be trade between them, which in turn will guarantee that price increments are uniquely determined.

Lemma 1 shows that equilibrium utility is increasing in the quality of the house consumed. The full support of incomes and the overlap of autarky utilities imply that the consumption levels y of households that consume a house of type k must be distributed with full support on some interval $[y_k, \bar{y}^k] \in [\underline{\theta}, \bar{\theta}]$. The unequivocally poorest and richest households do not have anyone to trade with, so $\underline{y}_0 = \underline{\theta}$ and $\bar{y}_N = \bar{\theta}$.

The wealthiest household that consumes a type- k house must be indifferent between trading up to the next house type. This indifference condition defines the price increments $\Delta_k = p_k - p_{k-1}$ as functions of upper bounds \bar{y}_k ,

$$\gamma(\bar{y}_k | x_k, x_{k+1}) = \{\Delta_{k+1} : u(x_k, \bar{y}_k) = u(x_{k+1}, \bar{y}_k - \Delta_{k+1})\} \text{ for all } k = 0, \dots, N-1. \quad (23)$$

Note that γ is single-valued for all $\bar{y}_k \in [\underline{\theta}, \bar{\theta}]$, with image in $(0, \bar{y}_k)$.

The same indifference conditions can also be described in terms of the lower bounds of the consumption intervals, with $u(x_k, \underline{y}_{k+1} + \Delta_{k+1}) = u(x_{k+1}, \underline{y}_{k+1})$, so

$$\underline{y}_{k+1} = \bar{y}_k - \Delta_{k+1} = \bar{y}_k - \gamma(\bar{y}_k | x_k, x_{k+1}). \quad (24)$$

Thus the lower bounds as well as the price increments are uniquely determined by the upper bounds, so the equilibrium allocation can be described in terms of the \bar{y}_k alone, where $k = 0, \dots, N - 1$.¹²

Demand for type- k houses is the sum of demands from each household type. Consider type- j households endowed with income θ . They will consume a type- k house if their wealth is in the same range as of those type- k households who consume their endowment: $p_j + \theta \in [p_k + \underline{y}_k, p_k + \bar{y}_k]$. The bounding inequalities for these intervals can be written as

$$\begin{aligned}\theta &\leq \bar{y}_k + p_k - p_j \\ \theta &\geq \underline{y}_k + p_k - p_j = \bar{y}_{k-1} + p_{k-1} - p_j\end{aligned}\tag{25}$$

for $k = 1, \dots, N$. (Recall that $\bar{y}_N = \bar{\theta}$.) Total demand for type- k houses is

$$\begin{aligned}Q_k(\bar{y}) &= \sum_{j=0}^N [F_j(\bar{y}_k + p_k - p_j) - F_j(\bar{y}_{k-1} + p_{k-1} - p_j)] \\ &= \sum_{j=0}^{k-1} [F_j(\bar{y}_k + (\Delta_{j+1} + \dots + \Delta_k)) - F_j(\bar{y}_{k-1} + (\Delta_{j+1} + \dots + \Delta_{k-1}))] \\ &\quad + F_k(\bar{y}_k) - F_j(\bar{y}_{k-1} - \Delta_k) \\ &\quad + \sum_{j=k+1}^N [F_j(\bar{y}_k - (\Delta_{k+1} + \dots + \Delta_j)) - F_j(\bar{y}_{k-1} - (\Delta_k + \dots + \Delta_j))],\end{aligned}\tag{26}$$

where Δ_k are functions of \bar{y} as defined by (23) and (24). Define the excess demand functions as

$$Z_k(\bar{y}) = \max\{0, Q_k(\bar{y})\} - m_k,\tag{28}$$

where the infeasible negative demands allowed by (26) are removed. The nonnegativity constraint is binding if the upper bound \bar{y}_k is below the lower bound \underline{y}_k implied by \bar{y}_{k-1} .

Each household has zero net demand for housing units, because not trading amounts to demanding your own house. Thus $Z_N(\bar{y}) = -\sum_{k=0}^{N-1} Z_k(\bar{y})$ and market equilibrium can be defined as

$$Z_k(\bar{y}) = 0 \text{ for } k = 0, \dots, N - 1,\tag{29}$$

or, in vector form, $Z(\bar{y}) = 0$. This is a system of N equations in N unknowns.

¹²Equivalently, the allocation can be described in terms of the lower bounds \underline{y}_k or prices Δ_k , where $k = 1, \dots, N$. This choice of independent variables is a matter of convenience.

Consider the best response of an imaginary "player" who attempts to minimize the absolute value of excess demand for type- k houses with the sole instrument of choosing the upper bound \bar{y}_k . (The top house type, where $\bar{y}_N = \bar{\theta}$ by construction, does not have a "player" representing it.) Use \bar{y}_{-k} to denote the vector \bar{y} without the k :th element. The best response function is

$$a_k(\bar{y}_{-k}) = \arg \min_{a \in [\underline{\theta}, \bar{\theta}]} |Z_k(a|\bar{y}_{-k})|. \quad (30)$$

It is straightforward to show that Z_k is continuous and increasing in a , but there may be a flat region. This happens when a is below the lower bound \underline{y}_k determined by \bar{y}_{-k} . Then demand for k -type houses is zero, and $Z_k(a|\bar{y}_{-k}) = -m_k$. However, the highest possible lower bound is strictly below $\bar{\theta}$, by (24) it is $\bar{\theta} - \gamma(\bar{\theta}|x_{k-1}, x_k)$, so Z_k must be strictly increasing in some interval $[\theta, \bar{\theta}]$. There are two possibilities

$$a_k(\bar{y}_{-k}) = \begin{cases} \bar{\theta}, & Z_k(\bar{\theta}|\bar{y}_{-k}) \leq 0 \\ \theta, & Z_k(\theta|\bar{y}_{-k}) = 0 \end{cases}, \quad (31)$$

where $\theta \in (\underline{\theta}, \bar{\theta})$. Thus a_k is single-valued but may not be able to eliminate excess supply for k . Hence $a(\bar{y}) = (a_0(\bar{y}_{-0}), \dots, a_{N-1}(\bar{y}_{-(N-1)}))$ defines a continuous vector-valued function from $[\underline{\theta}, \bar{\theta}]^N$ to itself. Thus, by Brouwer Fixed Point theorem, there exists a fixed point $y^* = a(y^*)$.

Next we prove, by contradiction, that a fixed point must be a market equilibrium. Suppose that $y^* = a(y^*)$ but $Z_k(y^*) < 0$ for some k , so $y_k^* = \bar{\theta}$. Now consider the highest such k , so that $Z_h(y^*) = 0$ for all higher types $h \in \{k+1, \dots, N-1\}$. Since $\underline{y}_{k+1} = \bar{\theta} - \gamma(\bar{\theta}|x_k, x_{k+1}) > \underline{\theta}$ (due to overlap of autarky utilities) at least some $k+1$ -types are trading down, while no one among types k or lower is trading up to houses above k . Hence $Z_h > 0$ for at least one $h < k$, a contradiction with (31). Intuitively, all demand for houses $k+1$ and higher would have to come from households endowed with those houses. Since some of them are trading down, there must be excess demand for some $h > k$.

Thus any fixed point of Z indeed defines an equilibrium allocation of our model. Finally, since γ is strictly increasing, the equilibrium allocation is associated with unique price increments Δ_k , so associated equilibrium prices p_k are unique up to an additive constant. (However, we have not shown that there is a unique equilibrium allocation).

Appendix B: Data and preference inference

We use income and house price data from the American Housing Survey (AHS). House price is based on the survey question where respondents were asked to estimate the current market value of their house. We consider only homeowners, which amounts to assuming that rental housing forms an entirely separate market.

The choice of MAs was determined by the availability of data. The most populous MAs are covered only in national-level data (the National Surveys), where house prices are top coded at a common national threshold. This results in a disproportionate censoring of observations in the largest MAs. In those MAs where the national data is not decimated by top coding it suffers from small sample size, which unfortunately renders the national surveys useless for our purposes. Mid-size cities are covered in separate "Metro surveys," which have been conducted at irregular intervals, and less affected by the top-coding problem because it is done at city-level. Our sample includes all MAs from which there is metro survey data both in 2007 (the most recent year) and 1998 (the second most recent year for the MAs surveyed in 2007).

	Permanent income		Income		House value		Rank correlation	
	1998	2007	1998	2007	1998	2007	1998	2007
Baltimore	0.16	0.17	0.37	0.40	0.27	0.28	0.38	0.48
Boston	0.13	0.17	0.36	0.44	0.25	0.23	0.48	0.46
Houston	0.18	0.20	0.39	0.41	0.30	0.27	0.53	0.52
Minneapolis	0.14	0.16	0.33	0.36	0.23	0.23	0.50	0.49
Tampa	0.19	0.23	0.41	0.52	0.29	0.31	0.52	0.48
Washington	0.14	0.17	0.33	0.39	0.25	0.23	0.50	0.49

Table B1. Summary statistics for inequality (Gini coefficient) of permanent income, income, and house value, and for the Spearman rank correlation between income and house value. "Permanent income" is income kernel smoothed relative to house price.

Table B2 shows for each MA the number of observations by year, the inferred value for the elasticity of substitution, and the minimized value of mean absolute percentage error (MAPE). The last row corresponds to the case where the elasticity parameter is restricted to be common for all MAs. The common preferences case results in elasticity of 0.50, which is in the middle of the MA-specific elasticities which are also relatively close to each other, ranging from 0.32 to 0.89. The measure of fit varies substantially over different MAs. The fit is clearly better in Boston, Houston, and Minneapolis than in Tampa or Washington.

	N_{98}	N_{07}	$1/(1 - \rho)$	MAPE
Baltimore	2306	1211	0.42	4.6
Boston	1960	1075	0.63	2.8
Houston	1963	1105	0.89	2.3
Minneapolis	2741	1524	0.48	2.6
Tampa	2106	1255	0.49	8.3
Washington	2356	1315	0.32	9.1
All	13432	7485	0.50	12.2

Table B2. Number of observations by year, estimated elasticity of substitution, and the minimized mean absolute percentage error (MAPE) for the predicted price in 2007, i.e., the mean value of $|\log(p^{07}(i)) - \log(\hat{p}^{07}(i|\rho))|$. Last row ("All") reports the case where the preference parameter (ρ) is restricted to be the same in all cities.

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