Supplementary material to "Uncertainty and energy saving investments"

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Abstract

In this note we provide the explicit solution to the simple of model of Section 2. Proposition 1 follows from this solution. This proof builds on the myopia result explained in Section 3 of the paper. We derive the stopping rule for a myopic investor when the aggregate capacity $k$ is taken as given, and from this we derive the equilibrium path $k = k(x)$ and its properties.

Define

$$\beta_1 = \frac{1}{2} - \frac{(r - \delta)}{\sigma^2} + \sqrt{\left[\frac{(r - \delta)}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2r}{\sigma^2}} > 1,$$

$$\beta_2 = \frac{1}{2} - \frac{(r - \delta)}{\sigma^2} - \sqrt{\left[\frac{(r - \delta)}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2r}{\sigma^2}} < 0.$$

**Lemma 1** Given the specification (1)-(4) in the main text, the optimal cut-off rule for a myopic investor as defined in Lemma 1 in the text is

$$x^m(k) = \begin{cases} 
\frac{\delta \beta_1 (B + C)}{r B (\beta_1 - 1)} (r I - \frac{AC}{B + C} + \frac{BC}{B + C} k) & \text{for } x \leq A - Bk \\
- \frac{\beta_1 (B + C) (A - Bk)}{(A - Bk)^{1 - \beta_2} B (\frac{r}{2} + \frac{1 - \beta_1}{2})} & \text{for } x > A - Bk.
\end{cases} \quad (1)$$

**Proof.** Given $k$, the revenue process for an existing new plant is defined by

\[ P(x; k) = \begin{cases} 
\frac{C (A - Bk)}{B + C} + \frac{B}{B + C} x, & \text{for } x \leq A - Bk \\
A - Bk, & \text{for } x > A - Bk
\end{cases} \]

where we use the definitions

\[ Q(k) = \frac{C (A - Bk)}{B + C}, \quad R = \frac{B}{B + C}. \]
The value of an existing plant, denoted by $V(x; k)$, satisfies the following ordinary differential equation:

$$
\frac{1}{2} \sigma^2 X^2 V''(x; k) + (r - \delta) x V'(x; k) - rV(x; k) + P(x; k) = 0,
$$

where $r$ is the discount rate, and $\delta = r - \alpha$. The general solution of the equation is

$$
V(x; k) = \begin{cases} 
V_0(x; k), & \text{for } x \leq A - Bk \\
V_+ (x; k), & \text{for } x > A - Bk 
\end{cases}
= \begin{cases} 
B_0^0 x^\beta_1 + B_0^2 x^\beta_2 + \frac{Q(k)}{r} + \frac{Rx}{\delta}, & \text{for } x \leq A - Bk \\
B_1^0 x^\beta_1 + B_2^+ x^\beta_2 + \frac{A - Bk}{r}, & \text{for } x > A - Bk.
\end{cases}
$$

where

The two boundary conditions $\lim_{x \to 0^+} V(x; k) = \frac{Q(k)}{r}$ and $\lim_{x \to \infty} V(x; k) = \frac{A - Bk}{r}$ imply that $B_2^0 = 0$ and $B_2^+ = 0$. The two remaining parameters would be easily solved by requiring that the first and second derivatives of the value functions match at $x = A - Bk$.

Denote the value of the option to install such a plant by $F(x; k)$. This must satisfy the following differential equation:

$$
\frac{1}{2} \sigma^2 X^2 F''(x; k) + (r - \delta) X F'(x; k) - rF(x; k) = 0,
$$

which has the general solution

$$
F(x; k) = C_1 x^{\beta_1} + C_2 x^{\beta_2}.
$$

The boundary condition $\lim_{x \to \infty} F(x; k) = 0$ implies that $C_2 = 0$. The problem is to find $C_1$ and the myopic investment threshold $x^m$. There are two possible cases that must be considered separately: (1) $x^m \leq A - Bk$, and (2) $x^m > A - Bk$.

The boundary conditions in case $x^m \leq A - Bk$ are (taking into account that $B_2^0 = 0$):

$$
C_1 x^{\beta_1} = B_0^0 x^{\beta_1} + \frac{Q(k)}{r} + \frac{Rx}{\delta} - I,
\beta_1 C_1 x^{\beta_1 - 1} = \beta_1 B_0^0 x^{\beta_1 - 1} + \frac{R}{\delta}.
$$

The ceiling $A - Bk$ is irrelevant in this case, and one can solve variable $C_1 - B_0^0$ instead of $C_1$. To see this, write these equations as

$$
(C_1 - B_0^0) x^{\beta_1} = \frac{Q(k)}{r} + \frac{Rx}{\delta} - I,
\beta_1 (C_1 - B_0^0) x^{\beta_1 - 1} = \frac{R}{\delta}.
$$
From these, we obtain the following linear relationship between \( x^m \) and \( k \):

\[
x^m = \frac{-\delta \beta_1 \left( \frac{Q(k)}{r} - I_r \right)}{R(\beta_1 - 1)} = \frac{\delta \beta_1 (B + C)}{rB(\beta_1 - 1)} \left( rI - \frac{AC}{B + C} + \frac{BC}{B + C}k \right).
\] (2)

The boundary conditions in case \( x^m > A - Bk \) are

\[
C_1 x^{\beta_1} = B_2^+ x^{\beta_2} + \frac{A - Bk}{r} - I,
\]

\[
\beta_1 C_1 x^{\beta_1 - 1} = \beta_2 B_2^+ x^{\beta_2 - 1}.
\]

This implies that the investment trigger is given by the non-linear equation:

\[
x^m = \left( -\frac{\beta_1 (B + C) \left( \frac{A - Bk}{r} - I \right)}{(A - Bk)^{1-\beta_2} B \left( \frac{\beta_1}{r} + \frac{1 - \beta_1}{\sigma} \right)} \right)^\frac{1}{\beta_2}.
\]

For the properties of the equilibrium it is enough to focus on the case \( x^m \leq A - Bk \). Let us now use the notation \( \hat{x} \) for the equilibrium investment trigger which is defined by the myopic trigger \( x^m(k) \). We can see from (1) that for \( x^m \leq A - Bk \), the myopic investment trigger \( x^m(k) \) defines the equilibrium capacity as a linear function of the current record \( \hat{x} \)

\[
k(\hat{x}) = \frac{r(\beta_1 - 1)}{\beta_1 \delta C} \hat{x} + \frac{AC - rI(B + C)}{BC}.
\]

Let us now explain the role of volatility for the equilibrium description to apply. Recall that \( \hat{x}^* \) is the equilibrium investment trigger at which \( \hat{x}^* = x^m = P = A - Bk^* \). Using the formula for \( x^m(k) \) as given in (2), we can solve \( k^* \) from

\[
\frac{\delta \beta_1 (B + C)}{rB(\beta_1 - 1)} \left( rI - \frac{AC}{B + C} + \frac{BC}{B + C}k^* \right) = A - Bk^*,
\] (3)

which gives

\[
k^* = \frac{\beta_1 (\delta AC + rAB - \delta rI(B + C)) - rAB}{B(\beta_1 (rB + \delta C) - rB)},
\]

\[
\hat{x}^* = \frac{rI\delta \beta_1 (B + C)}{\beta_1 \delta C + rB(\beta_1 - 1)}
\]

where the latter equation is obtained by evaluating \( x^m(k) \) at \( k^* \). Consider now \( k = 0 \) and the condition (3). The ratio \( \beta_1 / (1 - \beta_1) \) increases in \( \sigma \) monotonically so that the left-hand side of (3) exceeds the right-hand side even at \( k = 0 \). This would imply that the market must shut down before new entry can take place. There is therefore a unique \( \sigma^* \) such that equation (3) holds as equality when \( k = 0 \). For all \( \sigma < \sigma^* \) we can find a strictly positive value for \( k^* \) and thus for \( \hat{x}^* \).
The investment trigger in terms of output price is

$$P_H(\hat{x}) = \frac{C(A - Bk)}{B + C} + \frac{B}{B + C} \hat{x} = rI + \frac{\beta_1 B(\delta - r) + rB}{\beta_1 \delta(B + C)} \hat{x} \text{ for } x \leq \hat{x}^*.$$ 

We see that the price is increasing in $\hat{x}$, implying contraction of output for $x \leq \hat{x}^*$. The price trigger is

$$P_H(\hat{x}) = A - Bk(\hat{x}) \text{ for } x > \hat{x}^*,$$

which is decreasing in $\hat{x}$. The output thus expands for $x > \hat{x}^*$.

The peak price follows by direct substitution

$$P_H(\hat{x}^*) = \frac{\beta_1 \delta rI(B + C)}{\beta_1 (rB + \delta C)} - rB,$$

which is increasing in $\sigma$. When $C \to 0$, the myopic investment trigger approaches

$$x^m \to \frac{\delta \beta_1 B}{rB(\beta_1 - 1)} rI,$$

which is independent of $k$. Thus, once this trigger is reached, there is a discrete one-time jump in the capacity path. This completes the proof of the Proposition 1.