Delay and Information Aggregation in Stopping Games with Private Information

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Motivation

- Investment is typically characterized by:
  - Flexible timing
  - Irreversibility
  - Uncertainty
- Literature on real options stresses exogenous revelation of uncertainty
- We focus on endogenous revelation of uncertainty:
  - Information is out there, but dispersed
  - Players learn by observing each other's timing decisions
  - Timing game with pure informational externalities

Main findings

- Inefficient delays
- Clusters
- Aggregate randomness

Related literature:

- Herding literature
- Observational learning with endogenous timing:
  - Chamley and Gale (1994), Chamley (2004)
- Large common values auctions:
  - Pesendorfer and Swinkels (1997), Kremer (2002)
  - Bulow and Klemperer (1991)

Setup

- N players
- State of nature $\omega \in \Omega$ (finite set) with prior $P(\omega)$
- Player i chooses stopping time t_i and gets payoff $v_i(t_i, \omega) = v(t_i, \omega)$
- Conditional on ω, $v(t_i, \omega)$ is maximized at $t_i = \omega$
- Hence, ω is the first-best optimal stopping time for all players

Information and Payoffs

- We assume that the payoff function v is quasi-supermodular in (t_i, ω).
- Examples include:
  - Quadratic loss:

$$v_i(t_i, \omega) = -\beta(t_i - \omega)^2.$$

  - Discounted loss from early stopping:

$$v_i(t_i, \omega) = e^{r \max\{\omega, t_i\}} \cdot V - e^{-rt_i} \cdot C.$$

Information and Payoffs

- Player i observes privately a signal θ_i from a joint joint distribution $G(\theta, \omega)$ on $[\underline{\theta}, \overline{\theta}] \times \Omega$
- We assume continuous signals
- G is symmetric across i and conditional on ω, θ_i is independent of θ_j.
- Signals satisfy (MLRP)

Information and Payoffs

Assumption

For all i, $\theta' > \theta$, and $\omega' > \omega$,

$$\frac{g(\theta' \mid \omega')}{g(\theta \mid \omega')} > \frac{g(\theta' \mid \omega)}{g(\theta \mid \omega)}.$$

- We also assume boundedly informative individual signals:

Assumption

There is a constant $\kappa > 0$ such that

$$\forall \theta, \omega, \qquad \frac{1}{\kappa} > g(\theta \mid \omega) > \kappa.$$

- Players observe each others' actions, but not their signals

Multi-stage timing game

- It is convenient to use continuous time
- But, how to deal with the simultaneous actions / infinitely quick responses?
- We define the game as a multi-stage timing game, where every stopping decision starts a new stage
- It is possible to have many consequtive stages where a player (or many players) stops at time $t = 0$.
- Interpretation: players stop sequentially but at the same "real time"
- In other words, players are able to respond quickly to information obtained from others

More formally:

- $\Gamma\left(s^k\right)$ is the timing game of stage k, $k = 1, ..., K$
- It is a Bayesian game where each player chooses stopping time $\tau_i^k\left(\theta_i\right) \in \mathbf{R}_+ \cup \infty$
- Stage k ends at random time $t^k = \min_{i \in N \setminus Q^k} \tau_i^k\left(\theta_i\right)$ and the game moves to stage $k + 1$
- The total real time at the beginning of stage k is $T^k = \Sigma_{l=0}^{k-1} t^l$

Information at the beginning of the stage

- The state vector $s^k = \left[s_1^k, ..., s_N^k\right]$ summarizes the common information about the types of all players at the beginning of stage k
- We are interested in symmetric monotonic strategies
- With such strategies, the state variables are always of the form:

$$s_i^k = \begin{cases} [\underline{\theta}^{l_i}, \overline{\theta}^{l_i}] \text{ for a player that stopped in stage } l_i < k \\ [\overline{\theta}^k, \overline{\theta}] \text{ for a player that has not yet stopped at } k \end{cases}$$

Information during the stage

- Since the game remains in stage k only as long as no player stops, the choice of $\tau^k(\theta_i)$ is relevant only as far as

$$\tau^k(\theta_j) \geq \tau^k(\theta_i) \text{ for all } j.$$

- Conditional on her stopping choice being payoff relevant at instant t in stage k, player i knows that

$$\theta_j \geq \min\{\theta \,|\, \tau^k(\theta) \geq t\} \equiv \underline{\theta}^k(t).$$

- To include the information flowing during a stage, we introduce the state variable:

$$s^k(t) = (s_i^k(t)) = \begin{cases} s_i^k \text{ for } i \in Q^k, \\ s_i^k \cap \left[\underline{\theta}^k(t), \overline{\theta}\right] \text{ for } i \in \mathcal{N}^k \end{cases}.$$

Payoffs

- The payoff of i at the beginning of stage s^k is the expected payoff conditional on the information at hand
- Players choose strategies that maximize their expected payoffs at each contingency
- By equilibrium we mean a Perfect Bayesian Equilibrium
- We focus on symmetric strategies

Symmetric equilibrium

Theorem

The game has a unique symmetric equilibrium, where every player adopts at stage k the following strategy:

$$\tau_*^k(\theta) =$$

$$\min\left\{t \geq 0 \,\middle|\, \mathbb{E}\left[v(t,\omega) \,\middle|\, s^k(\theta)\right] \geq \mathbb{E}\left[v(t',\omega) \,\middle|\, s^k(\theta)\right] \text{ for all } t' \geq t\right\}. \tag{1}$$

- Stopping times are *myopically* optimal
- This property leads to a simple algorithm to calculate realized stopping moments from realized signals

- For each stage, equilibrium strategy defines a time dependent cutoff level $\theta^k(t)$:

$$\theta^k(t) \equiv \begin{cases} \underline{\theta}^k \text{ if } 0 \leq t < \tau_*^k\left(\underline{\theta}^k\right) \\ \overline{\theta} \text{ if } t > \tau_*^k\left(\overline{\theta}\right) \\ \max\left\{\theta \mid \tau_*^k(\theta) \leq t\right\} \text{ if } \tau_*^k\left(\underline{\theta}^k\right) \leq t \leq \tau_*^k\left(\overline{\theta}\right) \end{cases} . \qquad (2)$$

- This is the marginal type that stops at time t in equilibrium

Theorem

$\theta^k(t) : [0, \infty) \rightarrow \left[\underline{\theta}^k, \overline{\theta}\right]$ is continuous, (weakly) increasing, and along the path of the informative equilibrium $\theta^k(0) > \underline{\theta}^k$ for $k \geq 1$.

Characterization of equilibrium

- At $t = 0$, many players may stop simultaneously
- For $t > 0$, probability of many players stopping simultaneously is zero
- There may be many consequtive stages that end at $t = 0$
- "stopping waves" or "clusters"

Large game limit

- Let $N \to \infty$
- Let $T_N(\theta, \omega)$ denote the (random) moment when the player with signal θ stops in the informative equilibrium of the game with N players
- Let $T(\theta, \omega) = \lim_{N \to \infty} T_N(\theta, \omega)$
- Let $T^N(\omega)$ be the random time of last stopping in the game with N players
- $T(\omega) = \lim_{N \to \infty} T^N(\omega)$
- Let $F(t|\omega) = \Pr\{T(\omega) \leq t\}$ denote the probability distribution of "time of collapse"
- Let $f(t|\omega)$ denote the corresponding density function

Main Theorem

Theorem

In the informative equilibrium of the game, we have for all $\omega < \overline{\omega}$,

1. $suppf(t \mid \omega) = [\max\{t(\underline{\theta}), \omega\}, \overline{\omega}]$.
2. *For all* $\theta, \theta' \in (\underline{\theta}, \overline{\theta})$,

$$\lim_{N \to \infty} \Pr\{T_N(\omega, \theta) = T_N(\omega, \theta')\} = 1.$$

Main Theorem: Interpretation

- Interpretation: In large games almost all players stop at the same instant of real time, and stopping is delayed relative to first best optimum.
- This stopping time is random (even when pooled information is precise)
- If it is always optimal to stop at some finite point in time, then learning is asymptotically complete

More on Large Games

- Denote the n:th order statistic in the game with N players by

$$\widetilde{\theta}_n^N \equiv \min\left\{\theta \in \left[\underline{\theta}, \overline{\theta}\right] \mid \#\left\{i \in \mathcal{N} \mid \theta_i \leq \theta\right\} = n\right\}. \tag{3}$$

and let

$$Y_n^N \equiv \left(\widetilde{\theta}_n^N - \underline{\theta}\right) \cdot N, \tag{4}$$

and

$$\Delta Y_n^N \equiv Y_n^N - Y_{n-1}^N = \left(\widetilde{\theta}_n^N - \widetilde{\theta}_{n-1}^N\right) \cdot N, \tag{5}$$

where by convention we let $\theta_0^N \equiv \underline{\theta}$ and $Y_0^N \equiv 0$.

Theorem

Fix $n \in \mathbb{N}_+$ and denote by $[\Delta Y_1^\infty, ..., \Delta Y_n^\infty]$ a vector of n independent exponentially distributed random variables with parameter $g(\underline{\theta} \mid \omega)$:

$$\Pr(\Delta Y_1^\infty \leq x_1, ..., \Delta Y_n^\infty \leq x_n) = e^{-g(\underline{\theta}|\omega)\cdot x_1} \cdot ... \cdot e^{-g(\underline{\theta}|\omega)\cdot x_n}.$$

Consider the sequence of random variables $\left\{\left[\Delta Y_1^N, ..., \Delta Y_n^N\right]\right\}_{N=n}^\infty$, where for each N the random variables ΔY_i^N are defined by (3) - (5). As $N \to \infty$, we have:

$$\left[\Delta Y_1^N, ..., \Delta Y_n^N\right] \xrightarrow{\mathcal{D}} [\Delta Y_1^\infty, ..., \Delta Y_n^\infty],$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution.

More on Large Games

Corollary

Y_n^N converges to a Gamma distributed random variable:

$$Y_n^N = \sum_{i=1}^{n} \Delta Y_i^N \xrightarrow{\mathcal{D}} \sum_{i=1}^{n} \Delta Y_i^\infty \equiv Y_n^\infty,$$

where $Y_n^\infty \sim Gamma\left(n, g\left(\underline{\theta} \mid \omega\right)\right)$.

Payoffs

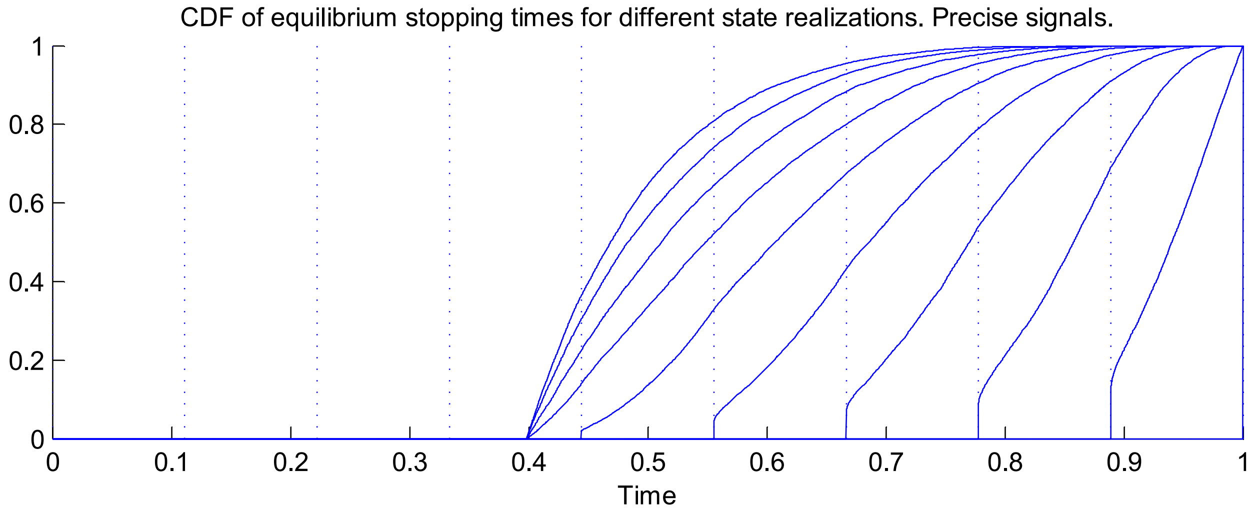
- Equilibrium determined by players with the lowest possible signals
- These players indifferent between acting upon own signal and moving last
- All other players benefit from observing the equilibrium outcome
- In the ex post sense, almost all players get the same payoff (part 1 of the Theorem)
- In the interim sense, expected payoffs are quite different (players with different signals predict different times of ending for the game).

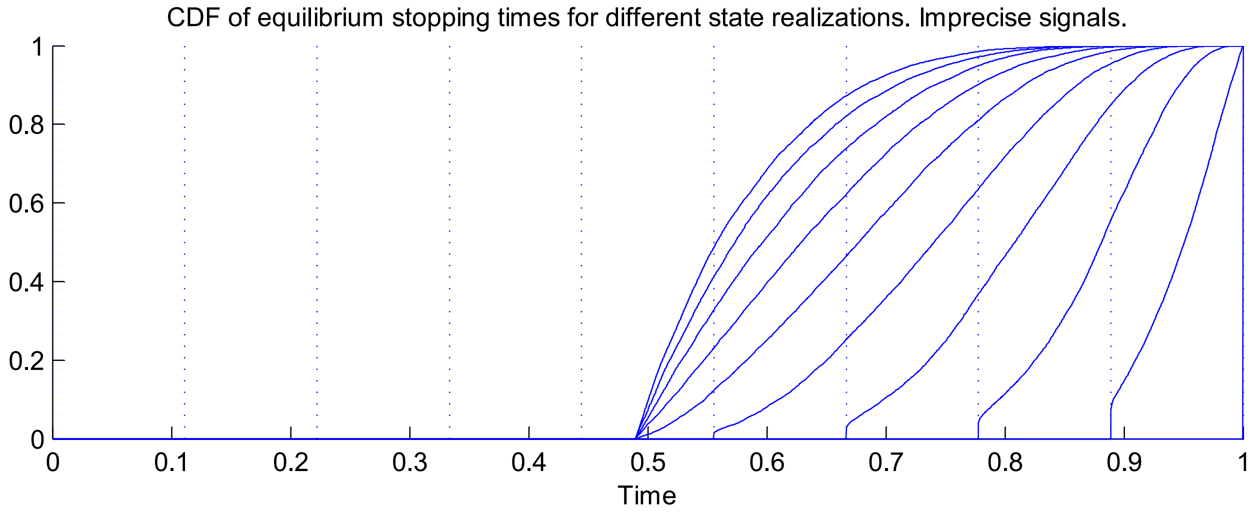
Example by Monte-Carlo simulation

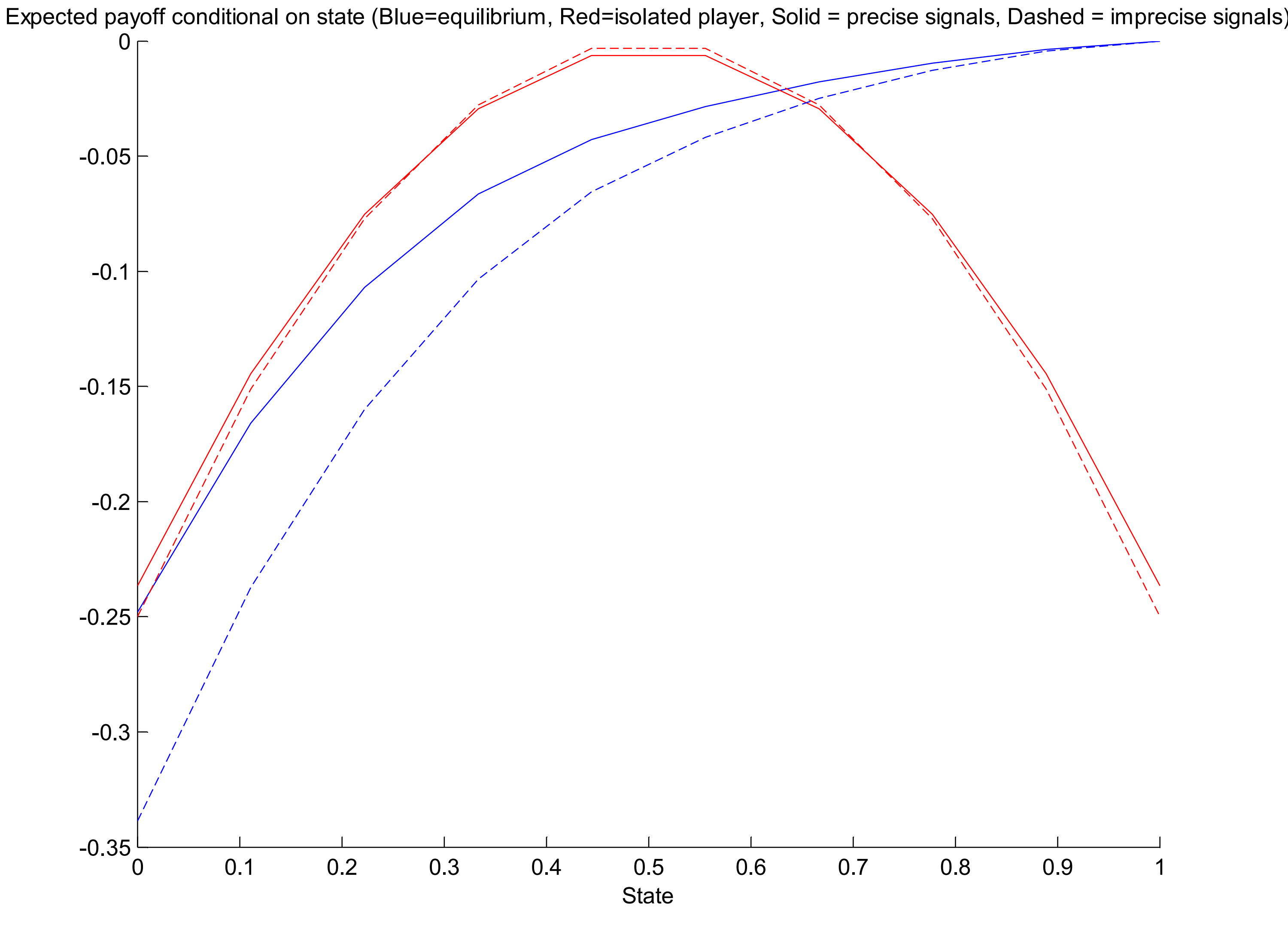
- Specify the model as follows:

$$\begin{aligned}
\Omega &= \left\{0, \frac{1}{S-1}, \frac{2}{S-1}, ..., \frac{S-2}{S-1}, 1\right\}, \\
\underline{\theta} &= 0, \overline{\theta} = 1, \\
g(\theta &\mid \omega) = 1 + \gamma\left(\omega - \frac{1}{2}\right)\left(\theta - \frac{1}{2}\right), \\
v\left(t, \omega\right) &= -\left(\omega - t\right)^2.
\end{aligned}$$

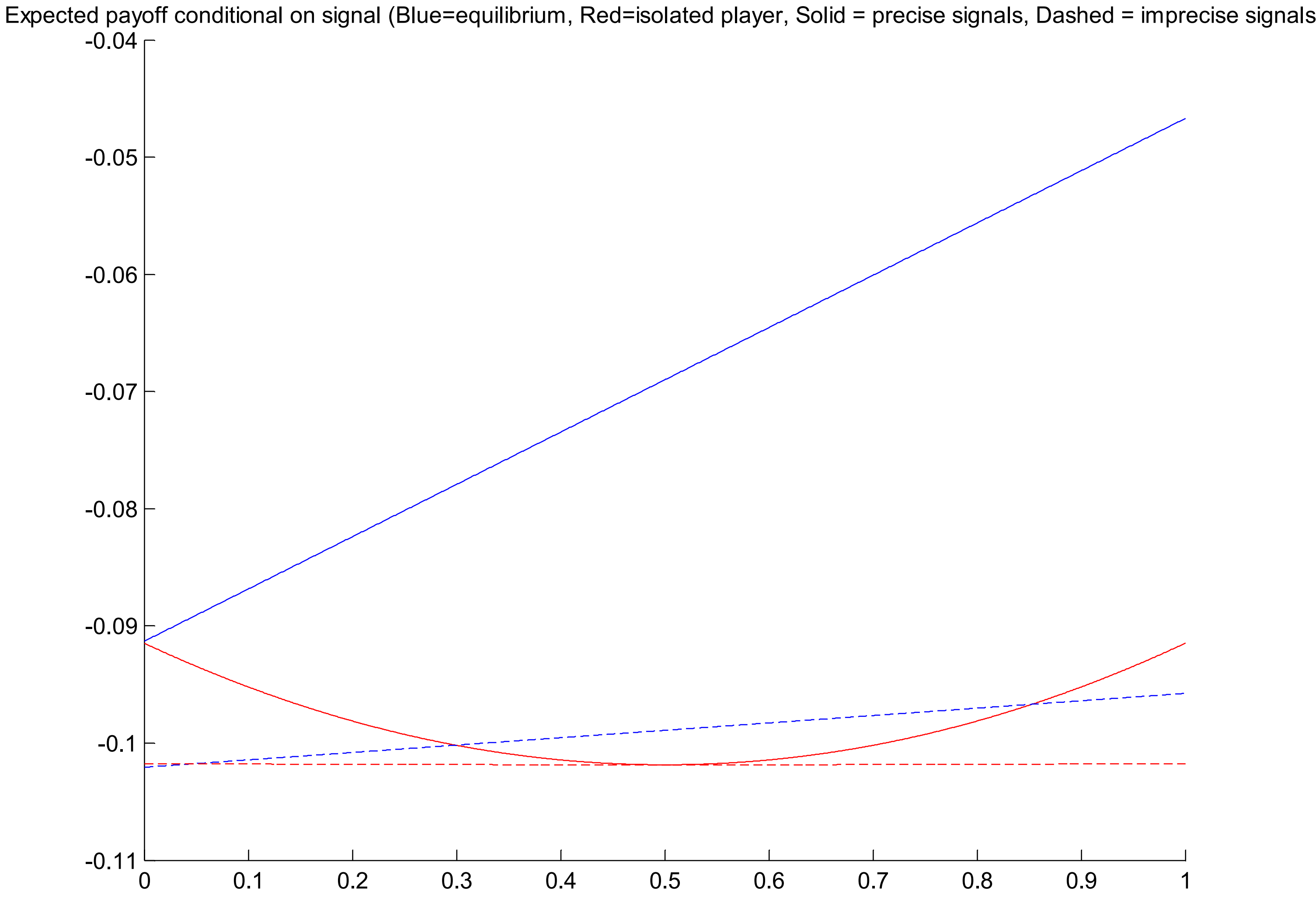
- Work directly in the large game limit ($N \to \infty$)


CDF of equilibrium stopping times for different state realizations. Precise signals.
0
0.2
0.4
0.6
0.8
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Time


CDF of equilibrium stopping times for different state realizations. Imprecise signals.
0
0.2
0.4
0.6
0.8
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Time



Expected payoff conditional on state (Blue=equilibrium, Red=isolated player, Solid = precise signals, Dashed = imprecise signals)
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
State


Expected payoff conditional on signal (Blue=equilibrium, Red=isolated player, Solid = precise signals, Dashed = imprecise signals
-0.04
-0.05
-0.06
-0.07
-0.08
-0.09
-0.1
-0.11
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Extensions

- Payoffs depend on own signals
- Payoffs depend on others' actions
  - Coordination
  - Congestion