Computational Appendix to "Uncertainty and energy saving investments"

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1 Discretizing the energy price process

Our starting point for the oil price process is the continuous time Geometric Brownian Motion:

$$dx = \mu x dt + \sigma x dz.$$

Letting Δ represent the real-time duration of one discrete time period, we can approximate the above process by a binomial process that governs the evolution of x_t as follows:

$$x_{t+1} = \begin{cases} x_t e^{\sigma\sqrt{\Delta}} & \text{with probability } q = \frac{1}{2}\left(1 + \frac{\mu - \frac{\sigma^2}{2}}{\sigma}\sqrt{\Delta}\right) \\ x_t e^{-\sigma\sqrt{\Delta}} & \text{with probability } 1-q \end{cases}.$$

The discount factor per period in the discrete version of the model is:

$$\delta \equiv e^{-r\Delta}.$$

In our numerical computations, we define a grid $\mathbb{X}$ consisting of possible values of energy price as follows:

$$\mathbb{X} = \left\{x^1, ..., x^M\right\},$$

where

$$x^m \equiv x^0 \cdot \left(e^{\sigma\sqrt{\Delta}}\right)^m, \; m = 1, ..., M.$$

Here, x^0 is some positive, arbitrarily small real number, and M is some positive, arbitrarily large integer. Within this truncated grid, we assume that the boundary grid points are absorbing states. By choosing a wide enough grid, the effect of this approximation can be made negligible. The Markov process used in the computations is then as follows. For $x_t = x^m$, $1 < m < M$:

$$x_{t+1} = \begin{cases} x^{m+1} & \text{with probability } q = \frac{1}{2}\left(1 + \frac{\mu - \frac{\sigma^2}{2}}{\sigma}\sqrt{\Delta}\right) \\ x^{m-1} & \text{with probability } 1-q \end{cases}, \tag{1}$$

and for $x_t = x^1$ and $x_t = x^M$:

$$x_{t+1} = x_t \text{ with probability 1.} \tag{2}$$

2 Evaluating the value functions used in computations

In our computations, we need to evaluate repeatedly expected values of the following type. Given two grid points $\underline{x}, \overline{x} \in \mathbb{X}$, some known function $v : \mathbb{X} \to \mathbb{R}$, and two known boundary values $V(\underline{x}), V(\overline{x}) \in \mathbb{R}$, define the following value function:

$$V(x) \equiv \mathbb{E}\left[\sum_{t=0}^{t^*} \delta^t v(x_t) + \delta^{t^*}(V(x_{t^*}))\right],\ x \in \mathbb{X} \cap [\underline{x}, \overline{x}], \tag{3}$$

where x_t is the Markov process defined in (1) - (2) s.t. $x_0 = x$, and where $t^* = \min\{t \geq 0 \,| x_t \in \{\underline{x}, \overline{x}\}\}$. It is easy to evaluate (3) using value function iteration. Note that the value function is simply a vector $[\underline{x}, ..., \overline{x}]$ of real numbers corresponding to grid points $\mathbb{X} \cap [\underline{x}, \overline{x}]$, with $V(\underline{x})$ and $V(\overline{x})$ fixed. A simple algorithm to find $V(x)$ proceeds as follows:

1. Set $i = 0$ and set some initial value V^i for the value function (with $V^i(\underline{x}) = V(\underline{x})$ and $V^i(\overline{x}) = V(\overline{x})$).

2. Compute $V^{i+1}$from:

   $$V^{i+1}(x_t) = v(x_t) + \delta \mathbb{E} V^i(x_{t+1})$$

   for all $x_t \in \mathbb{X} \cap [\underline{x}, \overline{x}]$

3. Set $i = i + 1$ and repeat step 2 until the value function has converged.

3 Solving the equilibrium capacities

As explained in the paper, the equilibrium is computed by solving the corresponding myopic investment problem for each capacity level. That is, we discretize the capacity space letting

$$\mathbb{K} \equiv \{0, \Delta k, 2\Delta k, ..., K\},$$

where Δk is the size of each capacity increment and K is chosen to be so large that it will never be reaced in equilibrium (profits would not justify further entry at this level of capacity even if energy price were in infinity).

Let us consider the discrete time version of equation (11) in the paper. The problem of the myopic investor at an arbitrary capacity level $k \in \mathbb{K}$, facing feed-in tariff at level τ, can be written as:

$$F_{\tau}^{m}(x,k) = \max_{x^{*} \in \mathbb{X}} \mathbb{E} \sum_{t=t^{*}}^{\infty} \delta^{t} \cdot \Delta \cdot \left[R_{\tau}(k, x_{t}) - rI\right], \tag{4}$$

where

$$t^{*} \equiv \min\left\{t \geq 0 \,|x_{t} \geq x^{*}\right\},$$

and where $R_{\tau}(k, x_t)$ is the revenue flow of a new capital unit with tariff τ, capacity k, and energy price x_t. The solution to (4) can be solved by standard dynamic programming methods. In our matlab implementation we utilize the following fact: since $R_{\tau}(k, x_t)$ is increasing in x_t, a necessary and sufficient condition for immediate investment being optimal at some $x \in \mathbb{X}$ is:

$$V^{0}(x) \equiv \mathbb{E} \sum_{t=0}^{t^{+}-1} \delta^{t} \cdot \Delta \cdot \left[R_{\tau}(k, x_{t}) - rI\right] > 0, \tag{5}$$

where $x_0 = x$ and $t^{+} \equiv \min\left\{t > 0 \,|x_t > x\right\}$. Therefore, to solve (4), we simply find the lowest $x \in \mathbb{X}$ such that $V^0(x) > 0$. The expression (5) for an arbitrary $x \in \mathbb{X}$ can be evaluated by the value function iteration explained in section 2 above by setting $\underline{x} = 0$, $\overline{x} = \min\left\{x' \in \mathbb{X} \,|x' > x\right\}$, $V(\underline{x}) = \frac{\Delta}{\delta}\left[R_{\tau}(k, \underline{x}) - rI\right]$, $V(\overline{x}) = 0$, and $v(x_t) = \Delta \cdot \left[R_{\tau}(k, x_t) - rI\right]$.

Let us denote the solution to (4) for given k and τ by $x_{\tau}^{*}(k)$. Having solved this for all $k \in \mathbb{K}$, the equilibrium capacity path $\mathbf{k}_{\tau} : \mathbb{X} \to \mathbb{K}$ is:

$$\mathbf{k}_{\tau}(\widehat{x}) = \max\left\{k \in \mathbb{K} \,|x_{\tau}^{*}(k) \leq \widehat{x}\right\}.$$

4 Computing the effects of the subsidy on consumer surplus and total welfare

We want to evaluate the difference in the producers' and consumers' surpluses between two equilibrium paths: $\mathbf{k}_{\tau}$ (the equilibrium capacity with tariff τ) and $\mathbf{k}_{0}$ (the corresponding equilibrium without any tariff). We evaluate these as present values computed at $t = 0$ with some initial value for energy price x_0.

Let $P(k, x)$ denote the market price of electricity, let $C(k, x)$ denote the cost of procuring the annual demand from market, and let $S_{\tau}(k, x)$ denote the annual total subsidy with tariff τ, given total capacity k and oil price x. Here we show that the effects of τ on consumers' and producers' surpluses can be computed without explicitly computing $S_{\tau}(k, x)$.

Since capacity along an equilibrium path depends on the historical maximum value of the process, we need to evaluate expected discounted sums of functions that depend on both the current and the historical maximum value of the energy price process. To evaluate such expected values, it is useful to decompose the sequence of future dates into blocks in such a way that the capacity stays constant during each block. Towards that end, let t^m denote the earliest moment at which x_t hits a grid point at or above x^m:

$$t^m \equiv \min\left\{t = 0, 1, ... \left| x_t \geq x^m \right.\right\}, \; m = 1, ..., M.$$

Let $m^0 \in \{1, ...M\}$ denote the index of the initial grid point, i.e. we have $x_0 = x^{m^0}$. Then, the sequence $\{t^m\}_{m=m^0}^M$ is the random sequence of future dates that contains only periods at which x_t hits new record values (note that $t^{m^0} = 0$). Let us also define $t^{M+1} = \infty$. Then, given any function $f(x, \widehat{x})$, we can decompose the expected discounted sum of $f(x, \widehat{x})$ as follows:

$$\begin{aligned}
\mathbb{E}\sum_{t=0}^{\infty}\delta^t f(x_t; \widehat{x}_t) &= \mathbb{E}\sum_{m=m^0}^{M}\sum_{t=t^m}^{t^{m+1}-1}\delta^t f(x_t; x^m) \\
&= \sum_{m=m^0}^{M}\mathbb{E}\delta^{t^m}\left[\sum_{t=t^m}^{t^{m+1}-1}\delta^{(t-t^m)} f(x_t; x^m)\right] \\
&= \sum_{m=m^0}^{M}\rho^m \cdot \mathbb{E}\left[\sum_{t=t^m}^{t^{m+1}-1}\delta^t f(x_t; x^m)\right], \qquad (6)
\end{aligned}$$

where the last line defines $\rho^m \equiv \mathbb{E}\delta^{t^m}$ and utilizes the fact that δ^{t^m} and x_t, $t > t^m$, are independent random variables (by Markov property of x_t). The formula (6) is useful, because it breaks the infinite sum into a sum of $M - m^0 + 1$ terms, each of which has constant $\widehat{x}_t = x^m$. Thus, each of those terms is a discounted sum of a function that depends on current x_t only, and can thus easily be computed by the value function iteration explained in section 2.

We can now apply this formula to compute the expected total value of subsidies that accrue to the investors due to tariff τ. Setting $f(x, \widehat{x}) = S_\tau(x, \mathbf{k}_\tau(\widehat{x}))$ in (6), we can write the expected present value of subsidies as:

$$\begin{aligned}
S_\tau^0 &\equiv \mathbb{E}\sum_{t=0}^{\infty}\delta^t \cdot \Delta \cdot S_\tau(x_t, \mathbf{k}_\tau(\widehat{x}_t)) \qquad (7) \\
&= \sum_{m=m^0}^{M}\rho^m \cdot \mathbb{E}\left[\sum_{t=t^m}^{t^{m+1}-1}\delta^t \cdot \Delta \cdot S_\tau(x_t, \mathbf{k}_\tau(x^m))\right].
\end{aligned}$$

Moreover, we can utilize the fact that $\mathbf{k}_\tau(\widehat{x})$ is an *equilibrium* capacity path. That is, free entry eliminates any profits to the entrants. In particular, this means that the total discounted revenue per unit of capacity between any two entry dates must correspond exactly to the investment cost expressed in flow units:

$$\mathbb{E}\left[\sum_{t=t^m}^{t^{m+1}-1}\delta^t \cdot \Delta \cdot \frac{S_t(x_t, \mathbf{k}_\tau(x^m))}{\mathbf{k}_\tau(x^m)}\right] + \mathbb{E}\left[\sum_{t=t^m}^{t^{m+1}-1}\delta^t \cdot \Delta \cdot P(x_t, \mathbf{k}_\tau(x^m))\right] = \mathbb{E}\left[\sum_{t=t^m}^{t^{m+1}-1}\delta^t \cdot \Delta \cdot rI\right],$$

where the left hand side sums subsidies (first term) and market revenue (second term), and right hand side gives the investment costs. Rearranging this, and substituting in (7) allows us to express the total discounted value of subsidies in terms of the price function $P(x,k)$:

$$S_\tau^0 = \sum_{m=m^0}^{M} \rho^m \cdot \mathbb{E}\left[\sum_{t=t^m}^{t^{m+1}-1} \delta^t \cdot \Delta \cdot (rI - P(x_t, \mathbf{k}_\tau(x^m))) \cdot \mathbf{k}_\tau(x^m)\right].$$

The total consumer cost with tariff τ is the sum of market price paid by the consumers and the subsidies paid to the investors, i.e.:

$$\begin{aligned} C_\tau &= \sum_{m=m^0}^{M} \rho^m \mathbb{E}\left[\sum_{t=t^m}^{t^{m+1}-1} \delta^t \cdot \Delta \cdot C(x_t, \mathbf{k}_\tau(x^m))\right] + S_\tau^0 \qquad (8) \\ &= \sum_{m=m^0}^{M} \rho^m \mathbb{E}\left[\sum_{t=t^m}^{t^{m+1}-1} \delta^t \cdot \Delta \cdot [C(x_t, \mathbf{k}_\tau(x^m)) + (rI - P(x_t, \mathbf{k}_\tau(x^m))) \cdot \mathbf{k}_\tau(x^m)]\right]. \end{aligned}$$

The expected value in (8) can be computed by the value function iteration of Section 2 by setting:[1]

$$\begin{aligned} x &= x^m,\ \underline{x} = 0,\ \overline{x} = x^{m+1}, \\ V(\underline{x}) &= \frac{\Delta}{\delta}\left[C(\underline{x}, \mathbf{k}_\tau(x^m)) + (rI - P(\underline{x}, \mathbf{k}_\tau(x^m))) \cdot \mathbf{k}_\tau(x^m)\right], \\ V(\overline{x}) &= 0, \\ v(x_t) &= \Delta \cdot \left[C(x_t, \mathbf{k}_\tau(x^m)) + (rI - P(x_t, \mathbf{k}_\tau(x^m))) \cdot \mathbf{k}_\tau(x^m)\right]. \end{aligned}$$

The change in consumer surplus due to tariff τ is

$$\Delta C_\tau \equiv C_0 - C_\tau,$$

where C_0 is (8) evaluated at $\tau = 0$.

Let us then consider the total social cost due to a distortionary tariff. Let k denote an arbitrary capacity level. For each $k > 0$, let $t_\tau(k)$ denote the period at which equilibrium capacity exceeds k for the first time under tariff τ:

$$t_\tau(k) \equiv \min\{t \geq 0 : \mathbf{k}_\tau(x_t) \geq k\}.$$

Consider then an infinitesimal increment to capacity at current capacity level k. The socially optimal moment to build that incremental unit without distortionary tariff is given by $t_0(k)$, yet in equilibrium with tariff τ this unit is built already at moment $t_\tau(k) \leq t_0(k)$. Since the flow social benefit of that incremental unit is given by $P(x_t, k)$ and the corresponding flow social cost is

[1] Also $\rho^m = \mathbb{E}\delta^{t^m}$ can be computed with the same algorithm setting $x = x_0$, $\underline{x} = 0$, $\overline{x} = x^m$, $V(\underline{x}) = 0$, $V(\overline{x}) = 1$, and $v(x_t) = 0$ for all $x_t < \overline{x}$.

given by rI, the expected social cost of building that particular unit at $t_\tau(k)$ instead of optimal time $t_0(k)$, as measured at moment $t_\tau(k)$, is given by

$$\mathbb{E}\left[\sum_{t=t_\tau(k)}^{t_0(k)} \delta^t \cdot \Delta \cdot (rI - P(x_t, k))\right].$$

Summing this over all incremental capacity units (i.e. integrating over k) and expressing everything in the present value units of time t_0 gives us the total social loss due to tariff τ:

$$\Delta C_\tau^{soc} = \int_{k=0}^{\infty} \mathbb{E}\delta^{t_\tau(k)} \cdot \mathbb{E}\left[\sum_{t=t_\tau(k)}^{t_0(k)} \delta^t \cdot \Delta \cdot (rI - P(x_t, k))\right] \cdot dk. \tag{9}$$

Again, the expected values in this expression can be evaluated with the value function iteration. Finally, since the social cost of the distortinary tariff must be the sum of changes in the consumers' and producers' surpluses, we get the change in the producers' profits simply as:

$$\Delta W_\tau = \Delta C_\tau^{soc} - \Delta C_\tau.$$