Online Appendix - not for publication

For the ease of readers, this supplementary material contains the detailed derivations of Lemmas 5, 6, 7, and 8. We keep the same numbering as in the main text to facilitate easier comparison.

Lemma 5. *For all (x, q) such that $q < 1$ and $x \leq x^E(q)$, function $g(x, q)$ in (9) is strictly positive, strictly increasing in x, and Lipschitz continuous. Furthermore, $g(x^E(q), q) > x^{E\prime}(q)$ for $q < 1$ and $\lim_{q \to 1} g(x^E(q), q) = x^{E\prime}(1)$.*

Lemma 6. *For all (x, q) with $q < q^*(x)$, we have*

$$V^*_{xx}(x, q) = V^*_{xx}(x, q^*(x)) = B(q^*(x))\, \Phi_{xx}(x, q^*(x)).$$

Lemma 7. *For all (x, q) with $q > q^*(x)$, we have*

$$V^*_q(x, q) + x v_H(q) + (1 - x)\, v_L(q) \leq 0.$$

Lemma 8. *For all $(x, q^*(x))$ with $q^*(x) > 0$, we have*

$$\frac{V^*(x, q^*(x))}{q^*(x)} + x v_H(q^*(x)) + (1 - x)\, v_L(q^*(x)) > 0.$$

Proof of Lemma 5. Taking the derivative of $g(x, q)$ with respect x gives:

$$\begin{aligned}
g_x(x, q) = -\Bigg[&\beta''(q)\Big(x^2(1-2x)(\beta(q)-1)^3 v_H(q)^2 - 2(1-x)x\beta(q)(\beta(q)-1) \\
&\times v_H(q)v_L(q)((1-2x)\beta(q) - x) + (1-x)^2(1-2x)\beta(q)^3 v_L(q)^2\Big) \\
&+ \beta'(q)\Big(2x^2(2x-1)(\beta(q)-1)^2 v_H(q)^2 \beta'(q) + (1-x)^2\beta(q)^2 v_L(q) \\
&\times \Big(2(1-2x)v_L(q)\beta'(q) - 2x(\beta(q)-1)v'_H(q) - (1-2x)\beta(q)v'_L(q)\Big) \\
&+ x v_H(q)\Big(4(1-x)v_L(q)\beta'(q)\left((1-2x)\beta(q)^2 + 2x\beta(q) + x\right) \\
&- x(\beta(q)-1)^2\Big((1-2x)(\beta(q)-1)v'_H(q) + 2(1-x)\beta(q)v'_L(q)\Big)\Big)\Big)\Bigg] / \\
&\Big[(x(\beta(q)-1)^2 v_H(q) + (1-x)(\beta(q))^2 v_L(q))^2 \beta'(q)\Big].
\end{aligned}$$

Both $g(x, q)$ and $g_x(x, q)$ are bounded if their denominators are bounded away from zero. We show that this is true if $q < 1$ and $x \leq x^E(q)$ by showing that it holds at $x = x^E$. First for the denominator of $g(x, q)$ we have:

$$x^E(q)(\beta(q)-1)^2 v_H(q) + (1 - x^E(q))(\beta(q))^2 v_L(q) < 0, \tag{22}$$

for all $q \in [0,1)$. Notice that the left-side is increasing in x and hence (22) implies the same inequality for all lower x. The condition (22) is equivalent with

$$\frac{-(\beta(q)-1)\beta(q)v_H(q)v_L(q)}{\beta(q)v_L(q)-(\beta(q)-1)v_H(q)} < 0$$

which is true because the numerator is positive (other terms are positive except $v_L(q) < 0$) and the denominator is negative. Together with $\beta'(q) < 0$, this implies that the denominator of g is strictly positive and bounded away from zero. We can also conclude that both g and g_x are bounded and continuous in both x and q for all (x,q) such that $q < 1$ and $x \leq x^E(q)$. Hence g is Lipschitz continuous for all $q < 1$.

To see that $g(x,q) > 0$, it is now enough to show that the numerator of (9) is strictly positive. First notice that the second term inside the brackets is always positive but the first term can be negative.[16] The first term is scaled by x, while the second therm is scaled by $(1-x)$. Therefore, if the numerator is positive at a belief above the boundary, it must be positive for the belief at the boundary as well. Since the decentralized belief, $x^E(q)$, is always above the fully optimal boundary, we can use it to show that the numerator is positive.

Plugging in $x^E(q)$ to the numerator of (9) and dividing by $x(1-x)$ gives:

$$\frac{\beta(q)v_L(q)\left(\beta'(q)\left(\beta(q)-1\right)v_H'(q)-\left(\left(\beta(q)-1\right)\beta''(q)-2\left(\beta'(q)\right)^2\right)v_H(q)\right)}{\beta(q)v_L(q)+(1-\beta(q))v_H(q)}$$
$$+\frac{\left(1-\beta(q)\right)v_H(q)\left(\beta'(q)\beta(q)v_L'(q)-\left(\beta(q)\beta''(q)-2\left(\beta'(q)\right)^2\right)v_L\left(q\right)\right)}{\beta\left(q\right)v_L\left(q\right)+\left(1-\beta\left(q\right)\right)v_H\left(q\right)}.$$

Since the denominator is negative ($v_L < 0$ and $\beta > 1$), this is proportional to

$$[v_H(q)v_L'(q)-v_H'(q)v_L(q)]\beta'(q)\beta(q)(\beta(q)-1)-2v_H(q)v_L(q)(\beta'(q))^2,$$

which is always positive because $v_H(q) > 0$ and $v_L(q), v_H'(q), v_L'(q) < 0$. Hence, $q(x,q) > 0$ for all $q \in [0,1)$ and $x \leq x^E(q)$.

Similar direct calculations show that $g_x > 0$ for all (x,q) such that $q < 1$ and $x \leq x^E(q)$.

Next, insert $x^E(q)$ to (dropping all dependencies) (9):

$$\begin{aligned} g(x^E(q),q) =& \frac{\frac{-\beta(1-\beta)v_Lv_H}{(\beta v_L+(1-\beta)v_H)^2}}{\frac{\beta'\beta(1-\beta)v_Lv_H}{\beta v_L+(1-\beta)v_H}}\Bigg(\frac{\beta'\beta(1-\beta)(v_Lv_H'-v_L'v_H)}{\beta v_L+(1-\beta)v_H} \\ &+\frac{\beta v_Lv_H(-2\beta'^2+(\beta-1)\beta'')}{\beta v_L+(1-\beta)v_H}+\frac{(\beta-1)v_Lv_H(-2\beta'^2+\beta\beta'')}{\beta v_L+(1-\beta)v_H}\Bigg) \\ =& \frac{v_H\left(2v_L\beta'-(\beta-1)\beta v_L'\right)+(\beta-1)\beta v_Lv_H'}{((\beta-1)v_H-\beta v_L)^2}. \end{aligned}$$

[16]This follows from $v_L(q) < 0$,$v_\omega'(q) < 0, \beta'(q) < 0, \beta(q) > 1$ and that $\beta(q)\beta''(q) > 2(\beta'(q))^2$.

The derivative of the decentralized policy x^E is

$$x^{E\prime}(q) = \frac{v_H\left(v_L\beta' - (\beta-1)\beta v_L'\right) + (\beta-1)\beta v_L v_H'}{((\beta-1)v_H - \beta v_L)^2}.$$

By subtracting $x^{E\prime}(q)$ from $g(x^E(q), q)$, we get

$$g(x^E(q), q) - x^{E\prime}(q) = \frac{\beta'(q)v_L(q)v_H(q)}{(\beta(q)v_L(q) + (1-\beta(q))v_H(q))^2}.$$

This expression is strictly positive for $q < 1$ and goes to zero as q goes to 1 (since $v_H(q) \to 0$). □

Proof of Lemma 6. Fixing some (x, q) such that $q < q^*(x)$, differentiating (17) twice with respect to x, and simplifying gives:

$$\begin{aligned} &V_{xx}^*(x,q) = V_{xx}^*(x, q^*(x)) + 2(q^*)'(x)\left(V_{xq}^*(x, q^*(x)) + v_H(q^*(x)) - v_L(q^*(x))\right) \\ &+ (q^*)''(x)\left(V_q^*(x, q^*(x)) + xv_H(q^*(x)) + (1-x)v_L(q^*(x))\right) \\ &+ \left((q^*)'(x)\right)^2\left(V_{qq}^*(x, q^*(x)) + xv_H'(q^*(x)) + (1-x)v_L'(q^*(x))\right). \end{aligned} \tag{23}$$

Noting that $q^*(x)$ is the inverse function of $x^*(q)$, the second term on the right-hand side vanishes by condition (20) and the third term vanishes by the condition (19). Let us look at the last term. First, since (19) holds along the boundary $(x, q^*(x))$, we can totally differentiate it with respect to x to get:

$$\begin{aligned} 0 \;=\; & V_{xq}^*(x, q^*(x)) + V_{qq}^*(x, q^*(x))(q^*)'(x) + v_H(q^*(x)) - v_L(q^*(x)) \\ & + [xv_H'(q^*(x)) + (1-x)v_L'(q^*(x))](q^*)'(x). \end{aligned}$$

Applying (20), several terms disappear and this reduces to

$$V_{qq}^*(x, q^*(x)) + xv_H'(q^*(x)) + (1-x)v_L'(q^*(x)) = 0.$$

The last term in (23) vanishes as well, and it follows that $V_{xx}^*(x,q) = V_{xx}^*(x, q^*(x))$. □

Proof of Lemma 7. If the claim is not true, there must be some x and $q > q^*(x)$ such that

$$V_q^*(x,q) + xv_H(q) + (1-x)v_L(q) > 0. \tag{24}$$

We show that this leads to a contradiction by showing that (24) implies $V_{xq}^*(x,q) + v_H(q) - v_L(q) > 0$, which further implies that (24) holds also for all beliefs in $[x, x^*(q)]$, including $V_q^*(x^*(q), q) + x^*(q)v_H(q) + (1 - x^*(q))v_L(q) > 0$, which contradicts (19).

It remains to show that (24) implies $V_{xq}^*(x,q) + v_H(q) - v_L(q) > 0$. First notice that $V_q^*(x,q) = B_q(q)\Phi(x,q) + B(q)\Phi_q(x,q)$, which then together with (24) implies

$$B_q > -\frac{\Phi_q}{\Phi}B - \frac{xv_H + (1-x)v_L}{\Phi}$$

where we have left out all dependencies to simplify notation. We now get the following lower bound:

$$V_{xq}^* + v_H - v_L = B_q\Phi_x + B\Phi_{xq} + v_H - v_L > -\frac{\Phi_q\Phi_x}{\Phi}B - \frac{\Phi_x}{\Phi}(xv_H + (1-x)v_L)$$
$$+ B\Phi_{xq} + v_H - v_L = \Phi^{-1}[B(\Phi_{xq}\Phi - \Phi_q\Phi_x) + \Phi(v_H - v_L) - \Phi_x(xv_H + (1-x)v_L)]. \tag{25}$$

The first term can be simplified as

$$\Phi^{-1}B(\Phi_{xq}\Phi - \Phi_q\Phi_x) = \frac{B\Phi\beta'}{x(1-x)} = \frac{\Phi\beta'}{x(1-x)}\frac{\Phi_x^*(x^*v_H + (1-x^*)v_L) - \Phi^*(v_H - v_L)}{\Phi_{xq}^*\Phi^* - \Phi_q^*\Phi_x^*}$$
$$= \frac{x^*(1-x^*)}{x(1-x)}\frac{\Phi}{\Phi^*\Phi^*}[\Phi_x^*(x^*v_H + (1-x^*)v_L) - \Phi^*(v_H - v_L)],$$

where the notation Φ^* refers to $\Phi(x^*(q), q)$.

Now, (25) becomes

$$\frac{x^*(1-x^*)}{x(1-x)}\frac{\Phi}{\Phi^*\Phi^*}[\Phi_x^*(x^*v_H + (1-x^*)v_L) - \Phi^*(v_H - v_L)] \tag{26}$$
$$-\frac{1}{\Phi}[\Phi_x(xv_H + (1-x)v_L) - \Phi(v_H - v_L)]$$
$$= \frac{1}{x(1-x)}\left(\frac{\Phi}{\Phi^*}((\beta-1)x^*v_H + \beta(1-x^*)v_L) - ((\beta-1)xv_H + \beta(1-x)v_L)\right),$$

where we have used the following for both terms inside the brackets:

$$\Phi(v_H - v_L) - \Phi_x(xv_H + (1-x)v_L) = \Phi(v_H - v_L) - \Phi\frac{\beta - x}{x(1-x)}(xv_H + (1-x)v_L)$$
$$= \frac{-\Phi}{x(1-x)}((\beta-1)xv_H + \beta(1-x)v_L).$$

To conclude that (26) is larger than 0, notice first that $(\beta-1)xv_H + \beta(1-x)v_L < 0$ whenever $x < x^E(q)$ and that it is increasing in x. Then observe that $\Phi/\Phi^* \in (0,1)$ and hence $(\beta-1)xv_H + \beta(1-x)v_L < (\Phi/\Phi^*)((\beta-1)x^*v_H + \beta(1-x^*)v_L)$.

We conclude that $V_q^* + xv_H + (1-x)v_L > 0$ implies $V_{xq}^* + v_H - v_L > 0$ and the proof is complete. □

Proof of Lemma 8. By definition of function $\Phi(x,q)$, the following holds for all $x > 0$, $q > 0$:

$$rB(q)\Phi(x,q) = \frac{1}{2}B(q)\Phi_{xx}(x,q)\frac{x^2(1-x)^2}{\sigma^2}q.$$

Differentiating w.r.t. q, the following holds as well:

$$r\left(B_q(q)\Phi(x,q)+B(q)\Phi_q(x,q)\right)=\frac{1}{2}B(q)\Phi_{xx}(x,q)\frac{x^2(1-x)^2}{\sigma^2}$$
$$+\frac{1}{2}\left(B_q(q)\Phi_{xx}(x,q)+B(q)\Phi_{xxq}(x,q)\right)\frac{x^2(1-x)^2}{\sigma^2}q$$
$$=r\frac{B(q)\Phi(x,q)}{q}+\frac{1}{2}\left(B_q(q)\Phi_{xx}(x,q)+B(q)\Phi_{xxq}(x,q)\right)\frac{x^2(1-x)^2}{\sigma^2}q.$$

In particular, this holds for any $q>0$, $x=x^*(q)$:

$$r\left(B_q(q)\Phi(x^*(q),q)+B(q)\Phi_q(x^*(q),q)\right)=r\frac{B(q)\Phi(x^*(q),q)}{q}$$
$$+\frac{1}{2}\left(B_q(q)\Phi_{xx}(x^*(q),q)+B(q)\Phi_{xxq}(x^*(q),q)\right)\frac{x^*(q)^2(1-x^*(q))^2}{\sigma^2}q. \quad (27)$$

From (19), we have

$$r\left(x^*(q)v_H(q)+(1-x^*(q))v_L(q)\right)+r\left(B_q(q)\Phi(x^*(q),q)+B(q)\Phi_q(x^*(q),q)\right)=0, \quad (28)$$

and so combining (27) and (28) we get

$$r\left(x^*(q)v_H(q)+(1-x^*(q))v_L(q)\right)+r\frac{B(q)\Phi(x^*(q),q)}{q} \quad (29)$$
$$+\frac{1}{2}\left(B_q(q)\Phi_{xx}(x^*(q),q)+B(q)\Phi_{xxq}(x^*(q),q)\right)\cdot\frac{x^*(q)^2(1-x^*(q))^2}{\sigma^2}q=0.$$

Plugging in (14) and (15) for $B(q)$ and $B_q(q)$, we get by direct computation at $x=x^*(q)$:

$$\begin{aligned}&B_q(q)\Phi_{xx}(x^*(q),q)+B(q)\Phi_{xxq}(x^*(q),q)\\=\;&\frac{x^*(q)(\beta(q)-1)^2v_H(q)-(1-x^*(q))(\beta(q))^2v_L(q)}{x^*(q)^2(1-x^*(q))^2}.\end{aligned} \quad (30)$$

Rearranging the equation that defines the policy function $x^E(q)$ of the decentralized equilibrium in Proposition 1, we have

$$x^E(q)(\beta(q)-1)v_H(q)-\left(1-x^E(q)\right)\beta(q)v_L(q)=0.$$

We have shown in Part 1 of the Appendix C.2 that $x^*(q)<x^E(q)$. Noting that $\beta(q)>1$, $v_H(q)>0$ and $v_L(q)<0$, it follows that

$$x^*(q)(\beta(q)-1)^2v_H(q)-(1-x^*(q))(\beta(q))^2v_L(q)<0$$

and so it follows from (30) that

$$B_q(q)\Phi_{xx}(x^*(q),q)+B(q)\Phi_{xxq}(x^*(q),q)<0. \quad (31)$$

Combining (29) and (31) gives

$$r\left(x^{*}\left(q\right)v_{H}\left(q\right)+\left(1-x^{*}\left(q\right)\right)v_{L}\left(q\right)\right)+r\frac{B\left(q\right)\Phi\left(x^{*}\left(q\right),q\right)}{q}>0,$$

which is equivalent to

$$xv_{H}\left(q^{*}\left(x\right)\right)+\left(1-x\right)v_{L}\left(q^{*}\left(x\right)\right)+\frac{B\left(q^{*}\left(x\right)\right)\Phi\left(x,q^{*}\left(x\right)\right)}{q^{*}\left(x\right)}>0$$

for all x for which $q^{*}\left(x\right)>0$. □