Online Appendix

for Gradual Learning from Incremental Actions

Learning process as the continuous limit

Consider a discrete model where the number of agents is n and where the period length is dt. Let the signal process be such that in each period, each agent who has stopped generates a normally distributed conditionally iid. signal:

$$y_t^i \sim N\left(\frac{\mu_\omega dt}{n}, \frac{\sigma^2 dt}{n}\right).$$

This normalization keeps the informativeness of the aggregate signal constant while letting the number of small agents to grow as in Bergemann and Välimäki (1997).

When the number of agents who has stopped is $k \leq n$, this implies the following aggregate signal:

$$\sum_{i=1}^{k} y_t^i \sim N\left(\mu_\omega dt\frac{k}{n}, \sigma^2 dt\frac{k}{n}\right).$$

Let $q = k/n$ denote the fraction of agents who have stopped. Now, the signal process (1) follows once we take the limit when $n \to \infty$ (and $k \to \infty$ so that k/n stays fixed) and $dt \to 0$.

Notice that the limiting distribution for the aggregate signal depends only on the mean and the variance of y_t^i (the central limit theorem). Hence, the signal process (1) is also the limiting process for the case where y_t^i is not normally distributed, including the case where agents communicate through binary signals.

Furthermore, we can rewrite the model so that the individual signals represent realized payoffs in a model where agents start receiving a stochastic flow payoff after stopping: $\pi_t(\theta) = \pi_\omega(\theta)\epsilon_t(\theta)$ where $\epsilon_t(\theta) \sim N\left(0, \sigma^2(\pi_H(\theta) - \pi_L(\theta))^2\right)$. The noise term is scaled so that every increment in q is equally informative. This assumption is not necessary: as Section 6 points out, both the analysis and the qualitative results remain unchanged with heterogeneous informativeness if the stopping profile is monotone. When we set $\pi_\omega(\theta) = rv_\omega(\theta)$, the expected stopping payoff is $x_t v_H(\theta)+(1-x_t)v_L(\theta)$ just like in the main text. Since there are no further actions after stopping, it does not matter how fast the agents learn privately the true state after they have stopped: the parameter σ can be interpreted to capture both the noise in the private learning and the noise in communication.

Social optimum: proof of Lemmas 4, 5, 6, and 7

Proof of Lemma 4. Taking the derivative of $g(x,q)$ with respect x gives:

$$
\begin{aligned}
g_x(x,q) = -\Big[&\beta''(q)\Big(x^2(1-2x)(\beta(q)-1)^3 v_H(q)^2 - 2(1-x)x\beta(q)(\beta(q)-1)\\
&\times v_H(q)v_L(q)((1-2x)\beta(q)-x) + (1-x)^2(1-2x)\beta(q)^3 v_L(q)^2\Big)\\
&+\beta'(q)\Big(2x^2(2x-1)(\beta(q)-1)^2 v_H(q)^2\beta'(q) + (1-x)^2\beta(q)^2 v_L(q)\\
&\times\Big(2(1-2x)v_L(q)\beta'(q) - 2x(\beta(q)-1)v_H'(q) - (1-2x)\beta(q)v_L'(q)\Big)\\
&+xv_H(q)\Big(4(1-x)v_L(q)\beta'(q)\left((1-2x)\beta(q)^2+2x\beta(q)+x\right)\\
&-x(\beta(q)-1)^2\Big((1-2x)(\beta(q)-1)v_H'(q)+2(1-x)\beta(q)v_L'(q)\Big)\Big)\Big)\Big]/\\
&\Big[(x(\beta(q)-1)^2 v_H(q) + (1-x)(\beta(q))^2 v_L(q))^2\beta'(q)\Big].
\end{aligned}
$$

Both $g(x,q)$ and $g_x(x,q)$ are bounded if their denominators are bounded away from zero. We show that this is true if $q<1$ and $x\le x^E(q)$ by showing that it hold at $x=x^E$:

$$
x^E(q)(\beta(q)-1)^2 v_H(q) + (1-x^E(q))(\beta(q))^2 v_L(q) < 0, \tag{33}
$$

for all $q\in[0,1)$. Notice that the left-side is increasing in x and hence (33) implies the same inequality for all lower x. The condition (33) is equivalent with

$$
\frac{\beta(q)(\beta(q)v_L(q)-(\beta(q)-1)v_H(q))}{\beta(q)^2 v_L(q)-(\beta(q)-1)^2 v_H(q)} > 1 \iff (\beta(q)-1)v_H(q) > 0,
$$

which holds as $\beta(q)>1$ and $v_H(q)>0$. We can conclude that g and g_x are bounded and continuous in both x and q for all (x,q) such that $q<1$ and $x\le x^E(q)$ and hence g is Lipschitz when we gap q away from 1. The denominator of $g(x,q)$ is strictly positive.

To see that $g(x,q)>0$, it is now enough to show that the numerator of (9) is strictly positive. First notice that the second term inside the brackets is always positive but the first term can be negative.[15] The first term is scaled by x, while the second therm is scaled by $(1-x)$. Therefore, if the numerator is positive at a belief above the boundary, it must be positive for the belief at the boundary as well. Since the decentralized belief, $x^E(q)$, is always above the fully optimal boundary, we can use it to show that the numerator is positive.

Plugging in $x^E(q)$ to the numerator of (9) and dividing by $x(1-x)$ gives:

$$\frac{\beta(q)v_L(q)\left(\beta'(q)\left(\beta(q)-1\right)v_H'(q)-\left(\left(\beta(q)-1\right)\beta''(q)-2\left(\beta'(q)\right)^2\right)v_H(q)\right)}{\beta(q)v_L(q)+(1-\beta(q))v_H(q)}$$
$$+\frac{\left(1-\beta(q)\right)v_H(q)\left(\beta'(q)\beta(q)v_L'(q)-\left(\beta(q)\beta''(q)-2\left(\beta'(q)\right)^2\right)v_L\left(q\right)\right)}{\beta\left(q\right)v_L\left(q\right)+\left(1-\beta\left(q\right)\right)v_H\left(q\right)}.$$

Since the denominator is negative ($v_L<0$ and $\beta>1$), this is proportional to

$$[v_H(q)v_L'(q)-v_H'(q)v_L(q)]\beta'(q)\beta(q)(\beta(q)-1)-2v_H(q)v_L(q)(\beta'(q))^2,$$

which is always positive because $v_H(q)>0$ and $v_L(q),v_H'(q),v_L'(q)<0$. Hence, $q(x,q)>0$ for all $q\in[0,1)$ and $x\leq x^E(q)$.

Similar direct calculations show that $g_x>0$ for all (x,q) such that $q<1$ and $x\leq x^E(q)$.

Next, insert $x^E(q)$ to (9):

$$\begin{aligned}g(x^E(q),q)=&\frac{\frac{-\beta(1-\beta)v_Lv_H}{(\beta v_L+(1-\beta)v_H)^2}}{\frac{\beta'\beta(1-\beta)v_Lv_H}{\beta v_L+(1-\beta)v_H}}\Bigg(\frac{\beta'\beta(1-\beta)(v_Lv_H'-v_L'v_H)}{\beta v_L+(1-\beta)v_H}\\&+\frac{\beta v_L\gamma_H(-2\beta'^2+(\beta-1)\beta'')}{\beta v_L+(1-\beta)v_H}+\frac{(\beta-1)v_Lv_H(-2\beta'^2+\beta\beta'')}{\beta v_L+(1-\beta)v_H}\Bigg)\\=&\frac{v_H\left(2v_L\beta'-(\beta-1)\beta\gamma_L'\right)+(\beta-1)\beta v_Lv_H'}{((\beta-1)v_H-\beta v_L)^2}.\end{aligned}$$

The derivative of the decentralized policy x^E is

$$x^{E\prime}(q)=\frac{v_H\left(v_L\beta'-(\beta-1)\beta v_L'\right)+(\beta-1)\beta v_Lv_H'}{((\beta-1)v_H-\beta v_L)^2}.$$

[15]This follows from $v_L(q)<0$,$v_\omega'(q)<0,\beta'(q)<0,\beta(q)>1$ and that $\beta(q)\beta''(q)>2(\beta'(q))^2$.

By subtracting $x^{E'}(q)$ from $g(x^E(q), q)$, we get

$$g(x^E(q), q) - x^{E'}(q) = \frac{\beta'(q) v_L(q) v_H(q)}{(\beta(q) v_L(q) + (1-\beta(q)) v_H(q))^2}.$$

This expression is strictly positive for $q < 1$ and goes to zero as q goes to 1 (since $v_H(q) \to 0$). □

Proof of Lemma 5. If the claim is not true, there must be some x and $q > q^*(x)$ such that

$$V_q(x, q) + x v_H(q) + (1-x) v_L(q) > 0. \tag{34}$$

We show that this leads to a contradiction by showing that (34) implies $V_{qx}(x, q) + v_H(q) - v_L(q) > 0$, which further implies that (34) holds also for all beliefs in $[x, x^*(q)]$, including $V_q(x^*(q), q) + x^*(q) v_H(q) + (1 - x^*(q)) v_L(q) > 0$, which contradicts the value matching condition (25).

It remains to show that (34) implies $V_{qx}(x, q) + v_H(q) - v_L(q) > 0$. First notice that $V_q(x, q) = B_q(q)\Phi(x, q) + B(q)\Phi_q(x, q)$, which then together with (34) implies

$$B_q > -\frac{\Phi_q}{\Phi} B - \frac{x v_H + (1-x) v_L}{\Phi}$$

where we have left out all dependencies to simplify notation. We now get the following lower bound:

$$\begin{aligned}
&V_{qx} + v_H - v_L = B_q \Phi_x + B\Phi_{qx} + v_H - v_L \\
&> -\frac{\Phi_q \Phi_x}{\Phi} B - \frac{\Phi_x}{\Phi}(x v_H + (1-x) v_L) + B\Phi_{qx} + v_H - v_L \\
&= \Phi^{-1}[B(\Phi_{qx}\Phi - \Phi_q \Phi_x) + \Phi(v_H - v_L) - \Phi_x(x v_H + (1-x) v_L)].
\end{aligned} \tag{35}$$

The first term can be simplified as

$$\Phi^{-1} B(\Phi_{qx}\Phi - \Phi_q\Phi_x) = \frac{B\Phi\beta'}{x(1-x)} = \frac{\Phi\beta'}{x(1-x)} \frac{\Phi^*_x(x^* v_H + (1-x^*) v_L) - \Phi^*(v_H - v_L)}{\Phi^*_{qx}\Phi^* - \Phi^*_q \Phi^*_x}$$

$$= \frac{x^*(1-x^*)}{x(1-x)} \frac{\Phi}{\Phi^*\Phi^*} [\Phi^*_x(x^* v_H + (1-x^*) v_L) - \Phi^*(v_H - v_L)],$$

where the notation Φ^* refers to $\Phi(x^*(q), q)$.

Now, (35) becomes

$$\frac{x^*(1-x^*)}{x(1-x)}\frac{\Phi}{\Phi^*\Phi^*}[\Phi_x^*(x^*v_H+(1-x^*)v_L)-\Phi^*(v_H-v_L)]$$
$$-\frac{1}{\Phi}[\Phi_x(xv_H+(1-x)v_L)-\Phi(v_H-v_L)]$$
$$=\frac{1}{x(1-x)}\Big(\frac{\Phi}{\Phi^*}((\beta-1)x^*v_H+\beta(1-x^*)v_L)-((\beta-1)xv_H+\beta(1-x)v_L)\Big), \tag{36}$$

where we have used the following for both terms inside the brackets:

$$\Phi(v_H-v_L)-\Phi_x(xv_H+(1-x)v_L)=\Phi(v_H-v_L)-\Phi\frac{\beta-x}{x(1-x)}(xv_H+(1-x)v_L)$$
$$=\frac{-\Phi}{x(1-x)}((\beta-1)xv_H+\beta(1-x)v_L).$$

To conclude that (36) is larger than 0, notice first that $(\beta-1)xv_H+\beta(1-x)v_L<0$ whenever $x<x^E(q)$ and that it is increasing in x. Then observe that $\Phi/\Phi^*\in(0,1)$ and hence $(\beta-1)xv_H+\beta(1-x)v_L<(\Phi/\Phi^*)((\beta-1)x^*v_H+\beta(1-x^*)v_L)$.

We conclude that $V_q+xv_H+(1-x)v_L>0$ implies $V_{qx}+v_H-v_L>0$ and the proof is complete. □

Proof of Lemma 6. Fixing some (x,q) such that $q<q^*(x)$, differentiating (23) twice with respect to x, and simplifying gives:

$$\begin{aligned} V_{xx}(x,q) &= V_{xx}(x,q^*(x)) \\ &\quad +2(q^*)'(x)\left(V_{qx}(x,q^*(x))+v_H(q^*(x))-v_L(q^*(x))\right) \\ &\quad +(q^*)''(x)\left(V_q(x,q^*(x))+xv_H'(q^*(x))+(1-x)v_L'(q^*(x))\right) \\ &\quad +\left((q^*)'(x)\right)^2\left(V_{qq}(x,q^*(x))+xv_H'(q^*(x))+(1-x)v_L'(q^*(x))\right). \end{aligned} \tag{37}$$

The second term on the right-hand side vanishes by the value-matching condition (25) and the third term vanishes by the smooth-pasting condition (26). Let us look at the last term. First, since (25) holds along the boundary $(x,q^*(x))$, we can totally differentiate it with respect to x to get:

$$\begin{aligned} 0 &= V_{qx}(x,q^*(x))+V_{qq}(x,q^*(x))(q^*)'(x)+v_H(q^*(x))-v_L(q^*(x)) \\ &\quad +[xv_H'(q^*(x))+(1-x)v_L'(q^*(x))](q^*)'(x). \end{aligned}$$

Applying (26), several terms disappear and this reduces to

$$V_{qq}\left(x,q^{*}\left(x\right)\right)+xv_{H}'\left(q^{*}\left(x\right)\right)+\left(1-x\right)v_{L}'\left(q^{*}\left(x\right)\right)=0.$$

The last term in (37) vanishes as well, and it follows that $V_{xx}\left(x,q\right)=V_{xx}\left(x,q^{*}\left(x\right)\right)$. Since this holds for any $q<q^{*}\left(x\right)$, it immediately implies that $V_{xxq}\left(x,q\right)=0$. □

Proof of Lemma 7. This is by direct computation. Recall that the value function for $q\geq q^{*}(x)$ is $V(x,q)=B(q)\Phi(x,q)$ and hence

$$V_{xxq}=B_{q}(q)\Phi_{xx}(x,q)+B(q)\Phi_{xxq}(x,q).$$

Plugging in the expressions for $B_q(q)$ and $B(q)$ from (20) and (21), multiplying by r, simplifying, and evaluating at $q=q^{*}(x)$ gives:

$$\begin{aligned} rV_{xxq}\left(x,q^{*}\left(x\right)\right) &= \frac{\left(\beta\left(q^{*}\left(x\right)\right)-1\right)^{2}+x\left(1-x\right)}{x^{2}\left(1-x\right)^{2}}v_{L}\left(q^{*}\left(x\right)\right) \\ &\quad -\frac{\beta\left(q^{*}\left(x\right)\right)\left(\beta\left(q^{*}\left(x\right)\right)-1\right)}{x^{2}\left(1-x\right)^{2}}\left(v_{H}\left(q^{*}\left(x\right)\right)-v_{L}\left(q^{*}\left(x\right)\right)\right). \end{aligned}$$

Noting that $\beta\left(q^{*}\left(x\right)\right)>1$, $v_{L}\left(q^{*}\left(x\right)\right)<0$, and $v_{H}\left(q^{*}\left(x\right)\right)-v_{L}\left(q^{*}\left(x\right)\right)>0$, it follows that $V_{xxq}\left(x,q^{*}\left(x\right)\right)<0$.

□

Static implementation: proof of Lemma 3

Proof. We first prove the increasing differences property that is crucial for global incentive compatibility. Denote by $U\left(\theta,\widetilde{\theta}\right)$ the expected payoff at time $t=0$ for type θ that reports $\widetilde{\theta}$:

$$U\left(\theta,\widetilde{\theta}\right)=\mathbb{E}\Big[e^{-r\tau(\widetilde{\theta})}(x_{\tau(\widetilde{\theta})}v_{H}(\theta)+(1-x_{\tau(\widetilde{\theta})})v_{L}(\theta))\Big]-P_{0}\left(\widetilde{\theta}\right).$$

Its partial derivative with respect to θ is

$$U_{1}\left(\theta,\widetilde{\theta}\right)=\mathbb{E}\Big[e^{-r\tau(\widetilde{\theta})}(x_{\tau(\widetilde{\theta})}v_{H}'(\theta)+(1-x_{\tau(\widetilde{\theta})})v_{L}'(\theta))\Big]. \tag{38}$$

The property that we want to prove is that $U_{1}\left(\theta,\widetilde{\theta}\right)$ is increasing in $\widetilde{\theta}$. To do that, note that applying the law of iterated expectations, we can write $U_{1}\left(\theta,\widetilde{\theta}\right)$

in terms of the initial belief x_0 as

$$\begin{aligned} U_1\left(\theta,\widetilde{\theta}\right) &= x_0\mathbb{E}\left(e^{-r\tau\left(\widetilde{\theta}\right)}v_H'\left(\theta\right)|\omega=H\right)+\left(1-x_0\right)\mathbb{E}\left(e^{-r\tau\left(\widetilde{\theta}\right)}v_L'\left(\theta\right)|\omega=L\right) \\ &= x_0v_H'\left(\theta\right)\mathbb{E}\left(e^{-r\tau\left(\widetilde{\theta}\right)}|\omega=H\right)+\left(1-x_0\right)v_L'\left(\theta\right)\mathbb{E}\left(e^{-r\tau\left(\widetilde{\theta}\right)}|\omega=L\right). \end{aligned}$$

Since $v_H'\left(\theta\right)\geq 0$ and $v_L'\left(\theta\right)\geq 0$ (with at least one of the inequalities strict), both terms in the above expression are positive. Reported type $\widetilde{\theta}$ enters the expression only through the discounting terms $\mathbb{E}\left(e^{-r\tau\left(\widetilde{\theta}\right)}|\omega\right)$. Since $\theta''>\theta'$ implies that $\tau\left(\theta''\right):=\inf\left\{t:q_t=1-F\left(\theta''\right)\right\}<\tau\left(\theta'\right):=\inf\left\{t:q_t=1-F\left(\theta'\right)\right\}$ with probability 1, it follows that $\mathbb{E}\left(e^{-r\tau\left(\widetilde{\theta}\right)}|\omega\right)$ is strictly increasing in $\widetilde{\theta}$ irrespective of state ω, and hence $U_1\left(\theta,\widetilde{\theta}\right)$ is strictly increasing in $\widetilde{\theta}$ as well.

We now utilize the above property to show that incentive compatibility holds for all types θ. Note first that we used the envlope theorem to set the transfer payment in (10) so that the value of θ satisfies:

$$\begin{aligned} W_0(\theta) &= \mathbb{E}\Big[e^{-r\tau(\theta)}(x_{\tau(\theta)}v_H(\theta)+(1-x_{\tau(\theta)})v_L(\theta))\Big]-P_0(\theta) \\ &= \mathbb{E}\left[\int_{\underline{\theta}}^{\theta}e^{-r\tau(s)}\Big(x_{\tau(s)}v_H'(s)+(1-x_{\tau(s)})v_L'(s))\Big)ds\right]. \end{aligned}$$

Therefore, for arbitrary θ' and θ'', we have

$$W_0(\theta'')-W_0(\theta')=\mathbb{E}\left[\int_{\theta'}^{\theta''}e^{-r\tau(s)}\Big(x_{\tau(s)}v_H'(s)+(1-x_{\tau(s)})v_L'(s))\Big)ds\right]. \tag{39}$$

We can now write the payoff of type θ who reports $\widetilde{\theta}$ as:

$$\begin{aligned} U\left(\theta,\widetilde{\theta}\right) &= U\left(\widetilde{\theta},\widetilde{\theta}\right)+\int_{\widetilde{\theta}}^{\theta}U_1\left(s,\widetilde{\theta}\right)ds=W_0\left(\widetilde{\theta}\right)+\int_{\widetilde{\theta}}^{\theta}U_1\left(s,\widetilde{\theta}\right)ds \\ &\leq W_0\left(\widetilde{\theta}\right)+\int_{\widetilde{\theta}}^{\theta}U_1\left(s,s\right)ds= \\ &= W_0\left(\widetilde{\theta}\right)+\mathbb{E}\left[\int_{\widetilde{\theta}}^{\theta}e^{-r\tau(s)}\Big(x_{\tau(s)}v_H'(s)+(1-x_{\tau(s)})v_L'(s))\Big)ds\right] \\ &= W_0\left(\theta\right), \end{aligned}$$

where the inequality uses the property that $U_1\left(\theta,\widetilde{\theta}\right)$ is increasing in $\widetilde{\theta}$, the second last equality uses (38), and the last equality uses (39). This shows that it is optimal for an arbitrary type θ to report truthfully.

To see that participation constraint is satisfied it sufficies to note that by reporting $\underline{\theta}$ any type gets a weakly positive payoff.

□

Dynamic implementation: proof of Proposition 4

Proof. Fix a boundary policy Q with a strictly increasing boundary $\widetilde{x}(q)$ and $\widetilde{x}(q_1) = 1$. We analyze the optimal stopping problem of arbitrary type θ. We will show that under $P^D(x)$ it is optimal to stop at the first hitting time of the point $(\widetilde{x}(q(\theta)), q(\theta))$ and the same conclusion holds under $P^S(q)$ if $\left(P^S\right)'(q) \geq 0$ for all $0 \leq q \leq q_1$. Finally, we show that if $\left(P^S\right)'(q) < 0$ for some q, then type $\theta(q)$ has a profitable deviation to stopping at some (x', q') with $x' < \widetilde{x}(q')$, $q' < q$.

Step 1: Optimal stopping when stopping below boundary prohibited

As a preliminary step we confirm that under both $P^D(x)$ and $P^S(q)$ it is optimal to stop at $(\widetilde{x}(q(\theta)), q(\theta))$ if stopping below the boundary is prohibited. This restricted stopping problem is still a Markovian stopping problem where the optimal solution is a first-hitting time of some of the boundary points. Under both $P^D(x)$ and $P^S(x)$, stopping at boundary point $(\widetilde{x}(q), q)$ entails transfer payment $P(\widetilde{x}(q), q)$ given in (11), which is designed in such a way that ex-ante expected payoff is equivalent to reporting $\theta(q)$ in the static direct mechanism. According to Lemma 3 it is optimal for θ to report truthfully in the static direct mechanism and so it follows that stopping at $(\widetilde{x}(q(\theta)), q(\theta))$ is optimal in the restricted stopping problem.

Step 2: Optimal stopping under dynamic posted price $P^D(x)$

We will utilize the property that the stopping value $u_\theta^D(x)$ is indepenent of q to show that even if stopping below the boundary is allowed, an agent will optimally stop at a first-hitting time of *some* boundary point. For contradiction, assume that type θ stops with a strictly positive probability at the first-hitting time of some point below the boundary and let (x', q') be the "left-most" such point. Formally, denoting by $F_\theta(x, q)$ the optimal value function of type θ, let (x', q'), $x' < \widetilde{x}(q')$, be a state point such that $F_\theta(x', q') = u_\theta^D(x')$ and $F_\theta(x, q) > u_\theta^D(x)$ for all points with $q < q'$.[16] We can write $F_\theta(x, q) = B_\theta(q)\,\Phi(x, q)$ for some function $B_\theta(q)$ so

[16] Note that we must have $F_\theta(\widetilde{x}(q), q) > u_\theta^D(\widetilde{x}(q))$ also at all boundary points with $q \leqq'$ because all those boundary points are visited before point (x', q') and hence otherwise stopping could not take place with positive probability at (x', q') .

that

$$\begin{aligned}\frac{\partial}{\partial q}F_\theta(x,q) &= B'_\theta(q)\,\Phi(x,q)+B_\theta(q)\,\Phi_q(x,q)\\ &= \Phi(x,q)\left[B'_\theta(q)+B_\theta(q)\,\beta'(q)\ln\left(\frac{x}{1-x}\right)\right],\end{aligned}$$

where we have used that $\Phi_q(x,q)=\beta'(q)\ln\left(\frac{x}{1-x}\right)\Phi(x,q)$.

Similar to the proof of Proposition 1, we note that the partial of a value function w.r.t. q must be zero at the boundary $x=\widetilde{x}(q)$, i.e. for $q<q'$ we have $\frac{\partial}{\partial q}F_\theta(x,q)_{|x=\widetilde{x}(q)}=0$ and so

$$B'_\theta(q)+B_\theta(q)\,\beta'(q)\ln\left(\frac{\widetilde{x}(q)}{1-\widetilde{x}(q)}\right)=0.$$

But since $\beta'(q)<0$ and $\widetilde{x}(q)>x$, this implies that

$$B'_\theta(q)+B_\theta(q)\,\beta'(q)\ln\left(\frac{x}{1-x}\right)>0,$$

and so we have $\frac{\partial}{\partial q}F_\theta(x,q)>0$. This is a contradiction with our assumption that $F_\theta(x',q')=u^D_\theta(x')$ and $F_\theta(x,q)>u^D_\theta(x)$ for all points with $q<q'$.

We can conclude that the optimal stopping time must be the first hitting time of some boundary point. This means that the solution must be the same as if stopping below the bounday is prohibited. By step 1 above, it is then optimal for θ to stop at the first-hitting time of point $(\widetilde{x}(q(\theta)),q(\theta))$.

Step 3: Optimal stopping under simple posted price $P^S(q)$

Since the stopping value $u^S_\theta(x,q)$ is now a function of q as well as x, the argument in the second step above does not hold and we will use a more direct approach. We will first directly show that if $P^S(q)$ is increasing everywhere, it is optimal for type θ to stop at the first hitting time of point $(\widetilde{x}(q(\theta)),q(\theta))$ that we denote by

$$\tau^*_\theta:=\inf\{t:(x_t,q_t)=(\widetilde{x}(q(\theta)),q(\theta))\}.$$

After that we will show that if $\left(P^S\right)'(q)<0$ for some q, then type $\theta(q)$ has a profitable deviation to stop at some (x',q') with $q'<q(\theta)$, $x'<\widetilde{x}(q')$.

Let us first investigate the implication of the condition $\left(P^S\right)'(q)\geq 0$. We can write

$$P^S(q)=x(q)\,v_H(\theta(q))+(1-x(q))\,v_L(\theta(q))-S(q),$$

where $S(q)$ is the information rent obtained by type $\theta(q)$ evaluated at the moment of stopping:

$$S(q) := \mathbb{E}\left[\int_{\underline{\theta}}^{\theta(q)} e^{-r\tau(s)}\left(x_{\tau(s)}v_H'(s) + \left(1 - x_{\tau(s)}\right)v_L'(s)\right)ds \,|\tilde{x}(q), q\right].$$

We next differentiate $P^S(q)$ with respect to q. Note first that for any $s < \theta(q)$ we can write

$$\mathbb{E}\left[e^{-r\tau(s)} \,|\tilde{x}(q), q\right] = A_s(q)\,\Phi(\tilde{x}(q), q) \tag{40}$$

for some function $A_s(q)$. Since q_t increases at the boundary, the partial derivative of (40) w.r.t. q must be zero there, i.e.: $\frac{\partial}{\partial q}\mathbb{E}\left[e^{-r\tau(s)} \,|\tilde{x}(q), q\right] = 0$. Therefore, the change of (40) when moving the initial point along the boundary is

$$\begin{aligned}
\frac{d}{dq}\mathbb{E}\left[e^{-r\tau(s)} \,|\tilde{x}(q), q\right] &= \frac{\partial}{\partial x}\mathbb{E}\left[e^{-r\tau(s)} \,|x, q\right]_{x=\tilde{x}(q)} \tilde{x}'(q) \\
&= A_s(q)\,\Phi_x(\tilde{x}(q), q)\,\tilde{x}'(q) \\
&= \frac{\Phi_x(\tilde{x}(q), q)}{\Phi(\tilde{x}(q), q)}\tilde{x}'(q)\,\mathbb{E}\left[e^{-r\tau(s)} \,|\tilde{x}(q), q\right].
\end{aligned} \tag{41}$$

Using this, we get:

$$\begin{aligned}
S'(q) = &\left(\tilde{x}(q)\,v_H'(\theta(q)) + (1 - \tilde{x}(q))\,v_L'(\theta(q))\right)\theta'(q) \\
&+ \int_{\underline{\theta}}^{\theta(q)} \frac{\Phi_x(\tilde{x}(q), q)}{\Phi(\tilde{x}(q), q)}\tilde{x}'(q)\,\mathbb{E}\left[e^{-r\tau(s)}\left(\tilde{x}_{\tau(s)}v_H'(s) + \left(1 - \tilde{x}_{\tau(s)}\right)v_L'(s)\right) |\tilde{x}(q), q\right] ds \\
= &\left(\tilde{x}(q)\,v_H'(\theta(q)) + (1 - \tilde{x}(q))\,v_L'(\theta(q))\right)\theta'(q) + \frac{\Phi_x(\tilde{x}(q), q)}{\Phi(\tilde{x}(q), q)}S(q)\,\tilde{x}'(q),
\end{aligned} \tag{42}$$

where the first term is from differentiation with respect to the integral bound and the second term is from differentiation with respect to the initial point $\tilde{x}(q)$ using (41). The derivative of $P^S(q)$ with respect to q is then

$$\begin{aligned}
\left(P^S\right)'(q) =& \tilde{x}'(q)\,(v_H(\theta(q)) - v_L(\theta(q))) \\
&+ [\tilde{x}(q)\,v_H'(\theta(q)) + (1 - \tilde{x}(q))\,v_L'(\theta(q))]\,\theta'(q) - S'(q) \\
=& \left[v_H(\theta(q)) - v_L(\theta(q)) - \frac{\Phi_x(\tilde{x}(q), q)}{\Phi(\tilde{x}(q), q)}S(q)\right]\tilde{x}'(q).
\end{aligned}$$

Hence, $\left(P^S\right)'(q) \geq 0$ is equivalent to

$$v_H(\theta(q)) - v_L(\theta(q)) \geq \frac{\Phi_x(\tilde{x}(q), q)}{\Phi(\tilde{x}(q), q)}S(q), \tag{43}$$

a condition that we will utilize below.

We now show that it cannot be optimal to stop below the boundary $\widetilde{x}(q)$ for $q < q(\theta)$. Fix (x,q) with $q < q(\theta)$ and $x < \widetilde{x}(q)$. We show that stopping at (x,q) is dominated by waiting until x hits $\widetilde{x}(q)$. Denote the value under that stopping rule by

$$\underline{F}_\theta(x,q) := \mathbb{E}\left[e^{-r\tau(\widetilde{x}(q))}u_\theta^S(x,q)\,|x,q\right],$$

where $\tau(\widetilde{x}(q)) = \inf\{t : x_t = \widetilde{x}(q)\}$. Writing $\underline{F}_\theta(x,q) := \underline{A}_\theta(q)\,\Phi(x,q)$ and solving $\underline{A}_\theta(q)$ from the boundary condition $\underline{F}_\theta(\widetilde{x}(q),q) = u_\theta^S(\widetilde{x}(q),q)$ gives us

$$\underline{F}_\theta(x,q) = \frac{\Phi(x,q)}{\Phi(\widetilde{x}(q),q)}u_\theta^S(\widetilde{x}(q),q).$$

We aim to show that below the boundary, i.e. for $x < \widetilde{x}(q)$, we have $\underline{F}_\theta(x,q) > u_\theta^S(x,q)$. Let us differentiate these functions with respect to x, and evaluate the derivative at the boundary:

$$\frac{\partial}{\partial x}\left[u_\theta(x,q)\right]_{x=\widetilde{x}(q)} = v_H(\theta) - v_L(\theta)$$

and

$$\begin{aligned}
&\frac{\partial}{\partial x}\left[\underline{F}_\theta(x,q)\right]_{x=\widetilde{x}(q)} = \frac{\Phi_x(\widetilde{x}(q),q)}{\Phi(\widetilde{x}(q),q)}u_\theta(\widetilde{x}(q),q)\\
&= \frac{\Phi_x(\widetilde{x}(q),q)}{\Phi(\widetilde{x}(q),q)}\left[\widetilde{x}(q)\left(v_H(\theta) - v_H(\theta(q))\right) + (1-\widetilde{x}(q))\left(v_L(\theta) - v_L(\theta(q))\right) + S(q)\right]\\
&\leq v_H(\theta)\frac{\beta(q)-\widetilde{x}(q)}{1-\widetilde{x}(q)} + v_L(\theta)\frac{\beta(q)-\widetilde{x}(q)}{\widetilde{x}(q)} - v_H(\theta(q))\frac{\beta(q)-1}{1-\widetilde{x}(q)} - v_L(\theta(q))\frac{\beta(q)}{\widetilde{x}(q)}\\
&= v_H(\theta) - v_L(\theta) + \left(\frac{\beta(q)-1}{1-\widetilde{x}(q)}\right)\left(v_H(\theta) - v_H(\theta(q))\right) + \frac{\beta(q)}{\widetilde{x}(q)}\left(v_L(\theta) - v_L(\theta(q))\right)\\
&< v_H(\theta) - v_L(\theta),
\end{aligned}$$

where the first inequality utilizes the fact that $(P^s)'(q) \geq 0$ is equivalent to (43) and the second inequality utilizes the fact that $q < q(\theta)$ implies that $v_H(\theta) - v_H(\theta(q)) \leq 0$ and $v_L(\theta) - v_L(\theta(q)) \leq 0$ with at least one of the inequalities strict It now follows that

$$\frac{\partial}{\partial x}\left[\underline{F}_\theta(x,q)\right]_{x=\widetilde{x}(q)} < \frac{\partial}{\partial x}\left[u_\theta^S(x,q)\right]_{x=\widetilde{x}(q)},$$

and since $\underline{F}_\theta(x,q)$ is strictly convex in x and positive for $x < \widetilde{x}(q)$ while $u_\theta^S(x,q)$ is linear in x, we have

$$\underline{F}_\theta(x,q) > u_\theta^S(x,q) \text{ for } x < \widetilde{x}(q),$$

and so stopping at (x,q) is strictly dominated by waiting until x hits $\widetilde{x}(q)$. Since (x,q) was arbitrarily chosen, it can never be optimal to stop strictly below the boundary for $q<q(\theta)$. But we know from Step 1 of the proof that stopping at point $(\widetilde{x}(q(\theta)),q(\theta))$ dominates stopping at other boundary points. This implies that it can never be optimal for θ to stop ealier than at τ_θ^*.

The above argument ruled out stopping below the boundary only for $q<q(\theta)$. It remains to show that θ cannot benefit from delaying stopping beyond time τ_θ^* to the hitting time of some (x,q) with $q>q(\theta)$, $x<\widetilde{x}(q)$. For that it sufficies to show that it is optimal to stop at all boundary points for $q\geq q(\theta)$, i.e. at all $(\widetilde{x}(q),q)$ for $q\geq q(\theta)$. The proof follows similar reasoning as the proof of Proposition 1.

Suppose, to the contrary, that there is some $(\widetilde{x}(q),q)$ such that $q\geq q(\theta)$ where it is not optimal to stop. At that point we therefore have $\widetilde{F}_\theta(\widetilde{x}(q),q)>u_\theta^S(\widetilde{x}(q),q)$, where $\widetilde{F}_\theta(x,q)$ is the value function under the optimal stopping rule (whatever that may be). We will show next that this implies that along the boundary, the rate of change in $\widetilde{F}_\theta(\widetilde{x}(q),q)$ is higher than in $u_\theta^S(\widetilde{x}(q),q)$:

$$\frac{d}{dq}\widetilde{F}_\theta(\widetilde{x}(q),q)>\frac{d}{dq}u_\theta^S(\widetilde{x}(q),q). \tag{44}$$

We prove this separately in two possible cases. First, suppose that it is optimal to stop for some (x',q), where $x'<\widetilde{x}(q)$. In that case $\widetilde{F}_\theta(x',q)=u_\theta^S(x',q)$. Since $\widetilde{F}_\theta(x,q)$ must be strictly convex in x whenever it is optimal to wait (i.e. when $\widetilde{F}_\theta(x,q)>u_\theta^S(x,q)$), whereas $u_\theta^S(x,q)$ is linear in x with slope $v_H(\theta)-v_L(\theta)$, we must have

$$\frac{\partial}{\partial x}\left[\widetilde{F}_\theta(x,q)\right]_{x=\widetilde{x}(q)}>v_H(\theta)-v_L(\theta). \tag{45}$$

Let us now compare the rates of change in $\widetilde{F}_\theta(\widetilde{x}(q),q)$ and $u_\theta^S(\widetilde{x}(q),q)$ along the boundary. We have

$$\begin{aligned}\frac{d}{dq}\widetilde{F}_\theta(\widetilde{x}(q),q) &= \frac{\partial}{\partial x}\left[\widetilde{F}_\theta(x,q)\right]_{x=\widetilde{x}(q)}\widetilde{x}'(q)+\frac{\partial}{\partial q}\widetilde{F}_\theta(\widetilde{x}(q),q)\\ &= \frac{\partial}{\partial x}\left[\widetilde{F}_\theta(\widetilde{x},q)\right]_{x=\widetilde{x}(q)}\widetilde{x}'(q),\end{aligned}$$

where we have again utilized the fact that the partial of a waiting value w.r.t q must vanish at the boundary where q is increased, i.e. $\frac{\partial}{\partial q}\widetilde{F}_\theta(\widetilde{x}(q),q)=0$.

The stopping value at (x, q) is

$$\begin{aligned} u_\theta^S(x,q) &= xv_H(\theta) + (1-x)v_L(\theta) - P^S(q) \\ &= xv_H(\theta) + (1-x)v_L(\theta) \\ &\quad -x(q)v_H(\theta(q)) - (1-x(q))v_L(\theta(q)) + S(q) \end{aligned}$$

and so at the boundary $x = \widetilde{x}(q)$ the stopping value is

$$\begin{aligned} u_\theta^S(\widetilde{x}(q), q) &= \widetilde{x}(q)(v_H(\theta) - v_H(\theta(q))) \\ &\quad + (1-\widetilde{x}(q))(v_L(\theta) - v_L(\theta(q))) + S(q) \end{aligned}$$

and we can compute its rate of change along the boundary as:

$$\begin{aligned} &\frac{d}{dq}u_\theta^S(\widetilde{x}(q), q) = \widetilde{x}'(q)(v_H(\theta) - v_H(\theta(q)) - v_L(\theta) + v_L(\theta(q))) \\ &- [\widetilde{x}(q)v_H'(\theta(q)) + (1-\widetilde{x}(q))v_L'(\theta(q))]\theta'(q) + S'(q) \\ &= \widetilde{x}'(q)\left[(v_H(\theta) - v_L(\theta)) - (v_H(\theta(q)) - v_L(\theta(q))) + \frac{\Phi_x(\widetilde{x}(q), q)}{\Phi(\widetilde{x}(q), q)}S(q)\right], \quad (46) \end{aligned}$$

where the latter equality uses (42). Using (45) and (43), we then see that (44) holds.

We move to the second case. Suppose that it is optimal to wait for all (x, q), where $x < \widetilde{x}(q)$, in which case $\widetilde{F}_\theta(x, q) > u_\theta^S(x, q)$ for all $x < \widetilde{x}(q)$ and $\widetilde{F}_\theta(0, q) = 0$. In that case, function $\widetilde{F}_\theta(x, q)$ must take the form

$$\widetilde{F}_\theta(x, q) = \widehat{A}_\theta(q)\Phi(x, q)$$

for some function $\widehat{A}_\theta(q)$. Our assumption $\widetilde{F}_\theta(\widetilde{x}(q), q) > u_\theta^S(\widetilde{x}(q), q)$ is equivalent to

$$\widehat{A}_\theta(q)\Phi(\widetilde{x}(q), q) > \widetilde{x}(q)(v_H(\theta) - v_H(\theta(q))) + (1-\widetilde{x}(q))(v_L(\theta) - v_L(\theta(q))) + S(q)$$

or

$$\widehat{A}_\theta(q) > \frac{\widetilde{x}(q)(v_H(\theta) - v_H(\theta(q))) + (1-\widetilde{x}(q))(v_L(\theta) - v_L(\theta(q))) + S(q)}{\Phi(\widetilde{x}(q), q)}.$$

This implies

$$
\begin{aligned}
&\frac{d}{dq}\widetilde{F}_\theta\left(\widetilde{x}\left(q\right),q\right) > \widetilde{x}'\left(q\right)\frac{\Phi_x\left(\widetilde{x}\left(q\right),q\right)}{\Phi\left(\widetilde{x}\left(q\right),q\right)} \\
&\times\left[\widetilde{x}\left(q\right)\left(v_H\left(\theta\right)-v_H\left(\theta\left(q\right)\right)\right)+\left(1-\widetilde{x}\left(q\right)\right)\left(v_L\left(\theta\right)-v_L\left(\theta\left(q\right)\right)\right)+S\left(q\right)\right] \\
&=\widetilde{x}'\left(q\right)\left[\frac{\beta\left(q\right)-\widetilde{x}\left(q\right)}{\left(1-\widetilde{x}\left(q\right)\right)}\left(v_H\left(\theta\right)-v_H\left(\theta\left(q\right)\right)\right)\right. \\
&\left.+\frac{\beta\left(q\right)-\widetilde{x}\left(q\right)}{\widetilde{x}\left(q\right)}\left(v_L\left(\theta\right)-v_L\left(\theta\left(q\right)\right)\right)+\frac{\Phi_x\left(\widetilde{x}\left(q\right),q\right)}{\Phi\left(\widetilde{x}\left(q\right),q\right)}S\left(q\right)\right].
\end{aligned}
$$

Noting that $q > q\left(\theta\right)$ means that $v_H\left(\theta\right)-v_H\left(\theta\left(q\right)\right)\geq 0$ and $v_L\left(\theta\right)-v_L\left(\theta\left(q\right)\right)\geq 0$, and comparing the expression above to (46), we note again that (44) holds.

Having proved that $\widetilde{F}_\theta\left(\widetilde{x}\left(q\right),q\right) > u_\theta\left(\widetilde{x}\left(q\right),q\right)$ implies (44), we note that this would imply that $\widetilde{F}_\theta\left(\widetilde{x}\left(q'\right),q'\right) > u_\theta\left(\widetilde{x}\left(q'\right),q'\right)$ along the boundary $\left(\widetilde{x}\left(q'\right),q'\right)$ for all $q\leq q'\leq q_1$, and so $\widetilde{F}_\theta\left(\widetilde{x}\left(q_1\right),q_1\right) > u_\theta\left(\widetilde{x}\left(q_1\right),q_1\right)$. This is a contradiction since we know that it must be optimal to stop at state point $\left(\widetilde{x}\left(q_1\right),q_1\right)$. We conclude that it is optimal to stop at all $\left(\widetilde{x}\left(q\right),q\right)$, $q\left(\theta\right)\leq q\leq q_1$. It now follows that the optimal stopping time for θ is τ_θ^*, i.e. the first hitting time of $\left(\widetilde{x}\left(q\left(\theta\right)\right),q\left(\theta\right)\right)$.

As a final point we note that our conclusion hinges critically on the assumption that $\left(P^S\right)'\left(q\right)\geq 0$ for all q. If, in contrast, $\left(P^S\right)'\left(q\right) < 0$ for some $q\in\left(0,q_1\right)$, then there is a profitable deviation for type $\theta=\theta\left(q\right)$ to stop ealier than at time τ_θ^*. To see this, note that $\left(P^S\right)'\left(q\right) < 0$ implies that for $\theta=\theta\left(q\right)$, we have

$$
v_H\left(\theta\left(q\right)\right)-v_L\left(\theta\left(q\right)\right) < \frac{\Phi_x\left(\widetilde{x}\left(q\right),q\right)}{\Phi\left(\widetilde{x}\left(q\right),q\right)}S\left(q\right). \tag{47}
$$

Consider then the value of type $\theta\left(q\right)$ who plans to stop at $\left(\widetilde{x}\left(q\right),q\right)$:

$$
F_{\theta(q)}\left(x,q\right)=\widetilde{A}_{\theta(q)}\left(q\right)\Phi\left(x;q\right).
$$

Since

$$
\begin{aligned}
&F_{\theta(q)}\left(\widetilde{x}\left(q\right),q\right)=u^S_{\theta(q)}\left(\widetilde{x}\left(q\right),q\right) \\
&=v_H\left(\theta\left(q\right)\right)-v_L\left(\theta\left(q\right)\right)-\left(v_H\left(\theta\left(q\right)\right)-v_L\left(\theta\left(q\right)\right)-S\left(q\right)\right)=S\left(q\right),
\end{aligned}
$$

we have

$$
\widetilde{A}_{\theta(q)}\left(q\right)=\frac{S\left(q\right)}{\Phi\left(\widetilde{x}\left(q\right),q\right)}
$$

and so

$$\frac{\partial}{\partial x}\left[F_{\theta(q)}\left(x,q\right)\right]_{x=\widetilde{x}(q)}=\frac{\Phi_{x}\left(\widetilde{x}\left(q\right),q\right)}{\Phi\left(\widetilde{x}\left(q\right),q\right)}S\left(q\right).$$

Noting that

$$\frac{\partial}{\partial x}\left[u_{\theta(q)}^{S}\left(x,q\right)\right]_{x=\widetilde{x}(q)}=v_{H}\left(\theta\left(q\right)\right)-v_{L}\left(\theta\left(q\right)\right),$$

equation (47) implies

$$\frac{\partial}{\partial x}\left[u_{\theta(q)}^{S}\left(x,q\right)\right]_{x=\widetilde{x}(q)}<\frac{\partial}{\partial x}\left[F_{\theta(q)}\left(x;q\right)\right]_{x=\widetilde{x}(q)}$$

and since $u_{\theta(q)}^{S}\left(\widetilde{x}\left(q\right),q\right)=F_{\theta(q)}\left(\widetilde{x}\left(q\right),q\right)$, it follows that for some $x<\widetilde{x}\left(q\right)$ we have

$$u_{\theta(q)}^{S}\left(x,q\right)>F_{\theta(q)}\left(x,q\right).$$

By continuity of F and u, this implies that there is some $q'<q$, $x'<\widetilde{x}\left(q'\right)$, such that $u_{\theta(q)}^{S}\left(x',q'\right)>F_{\theta(q)}\left(x',q'\right)$, and hence type $\theta\left(q\right)$ has a strictly beneficial deviation to stopping at that state, and that state is reached before τ_{θ}^{*} with a strictly positive probability.

We have now shown that $\widetilde{x}\left(q\right)$ can be implemented by the simple posted price $P^{s}\left(q\right)$ if an only if $\left(P^{s}\right)'\left(q\right)\geq 0$ for all $0\leq q\leq q_{1}$ and the proof is complete.

□

Revenue maximizing designer: proof of Proposition 5

Proof. Here, we show how to derive the virtual valuation representation for the designer's value. Suppose transfers follow an arbitrary policy, P_{τ}, adapted to $\mathcal{F}_{t}$. We denote the realization of P_{τ} at time t with p_{t}. The designer's expected revenue can be written as

$$\begin{aligned}&\int_{\underline{\theta}}^{\overline{\theta}}\mathbb{E}\Big[e^{-r\tau(\theta)}\Big(x_{\tau(\theta)}v_{H}(\theta)+(1-x_{\tau(\theta)})v_{L}(\theta)-W(\theta,x)\Big)\Big]f(\theta)d\theta\\&=\int_{\underline{\theta}}^{\overline{\theta}}\Big(\mathbb{E}\Big[e^{-r\tau(\theta)}\Big(x_{\tau(\theta)}v_{H}(\theta)+(1-x_{\tau(\theta)})v_{L}(\theta)\Big)\Big]f(\theta)d\theta\\&-\int_{\underline{\theta}}^{\overline{\theta}}\mathbb{E}\Big[\int_{\underline{\theta}}^{\theta(q)}e^{-r\tau(s)}\Big(x_{\tau(s)}v_{H}'(s)+(1-x_{\tau(s)})v_{L}'(s)\Big)ds\Big]f(\theta)d\theta,\end{aligned}$$

where we have used the envelope theorem for the agent's value (see Section 4 in the main text).

We can use Fubini's theorem to change the order of integration in the second term:

$$\begin{aligned}
&\int_{\underline{\theta}}^{\overline{\theta}} \mathbb{E}\Big[\int_{\underline{\theta}}^{\theta(q)} e^{-r\tau(s)}\Big(x_{\tau(s)}v_H'(s) + (1-x_{\tau(s)})v_L'(s)\Big)ds\Big] f(\theta)d\theta \\
=&\mathbb{E}\Big[\int_{\underline{\theta}}^{\overline{\theta}}\int_{s}^{\overline{\theta}} e^{-r\tau(s)}\Big(x_{\tau(s)}v_H'(s) + (1-x_{\tau(s)})v_L'(s)\Big) f(\theta)d\theta ds\Big] \\
=&\mathbb{E}\Big[\int_{\underline{\theta}}^{\overline{\theta}} e^{-r\tau(s)}\Big(x_{\tau(s)}v_H'(s) + (1-x_{\tau(s)})v_L'(s)\Big)(1-F(s))ds\Big].
\end{aligned}$$

The rest is simply to plug the above expression back into the designer's payoff and to write the integral over quantities rather than types (where we use $1 - F(\theta(q)) = q$). The profit maximizing designer's objective becomes

$$\mathbb{E}\Big[\int_0^1 e^{-r\tau(q)}(x_{\tau(q)}\phi_H(q) + (1-x_{\tau(q)})\phi_L(q))dq\Big],$$

where $\tau(q)$ is the stopping time of the q highest type buyer and $\phi(q)$ is his virtual valuation:

$$\phi_\omega(q) := v_\omega(\theta(q)) - v_\omega'(\theta(q))\frac{1-F(\theta(q))}{f(\theta(q))}.$$

□

Durable goods monopoly

Proposition 7. *There exist cutoffs $x_a \in (0,1)$ and $x_b \in (0,1)$ such that the monopoly quantity is larger if the initial belief is below x_a and the competitive market quantity is larger if the initial belief is above x_b.*

Proof. The existence of $x_b < 1$ follows from that the continuity of the policy functions and form that the complete information quantity is larger in the competitive market: $q^C(1) > q^M(1)$, where $q^C(1)$ solves $\theta(q^C(1)) = c$ and $q^M(1)$ solves $\theta(q^M(1)) - (1 - F(\theta(q^M(1))))/f(\theta(q^M(1))) = c$.

The existence of $x_a > 0$ follows from the same argument as that socially optimal policy is below the decentralized policy (details omitted): 1) To see that $x^M(0) \neq x^C(0)$, observe that the smooth pasting and value matching conditions for both the decentralized and the planner's policies cannot hold simultaneously when we approach $q \to 0$ along x^M. The reason why the same proof works for the monopolist's policy as for the planner's policy is that the monopolist's flow payoff is the same as the social planner's when $q = 0$. 2) To rule out $x^M(0) > x^C(0)$, notice that the monopolist gets strictly positive profits by selling to some small q whenever the initial belief is above $x^C(0) = x^{stat}(0)$. 3) Now, it is enough to use the continuity and monotonicity of the policy functions the same way as in the proof of Proposition 3 in the main text to conclude that there exists $x_a > 0$ such that the monopolists sells more for all beliefs below x_a.

□

Type-dependent informativeness: proof of Proposition 6

Proof. **Part (ii): fanatics.** Suppose $-i'(\theta)/i(\theta) \in [0, -v_L'(\theta)/v_L(\theta)]$.

First, we argue that the socially optimal stopping profile is monotone. We can use the same argument as in Lemma 2 but we need to normalize the informativeness of different agents. We show that for all agents $\theta, \theta' \in [\underline{\theta}, \overline{\theta}]$ such that $\theta > \theta'$ and for all realized stopping times $t, t' \in \mathbb{R}_+$ such that $t \leq t'$,

$$e^{-rt}\frac{v_\omega(\theta)}{i(\theta)} + e^{-rt'}\frac{v_\omega(\theta')}{i(\theta')} \geq e^{-rt'}\frac{v_\omega(\theta)}{i(\theta)} + e^{-rt}\frac{v_\omega(\theta')}{i(\theta')}.$$

. When the above condition holds, the planner always wants to implement any information stock process Z so that higher types stop first. The condition is equivalent to $(e^{-rt} - e^{-rt'})(v_\omega(\theta)/i(\theta) - v_\omega(\theta')/i(\theta')) \geq 0$. To see that this is satisfied, notice that $-i'(\theta)/i(\theta) \in [0, -v_L'(\theta)/v_L(\theta)]$ implies that $v_\omega(\theta)/i(\theta)$ is increasing in θ for both $\omega \in \{H, L\}$.

Now, we can use montonicity and define the planner's problem as finding the information stock policy Z that maximixes

$$\mathbb{E}\Big[\int_z^1 e^{-r\tau(s)}(x\hat{v}_H(s) + (1-x)\hat{v}_L(s))ds\Big|x, z; Z\Big], \tag{48}$$

where $\hat{v}_\omega(z) = v_\omega(h^{-1\prime}(z)))$. Notice that the problem is equivalent to (4).

All assumptions in Section 2 hold when $-i'(\theta)/i(\theta) \in [0, -v_L'(\theta)/v_L(\theta)]$ because then $\hat{v}_\omega(z)$ is decreasing in z for both $\omega \in \{H, L\}$, and hence the claim in the proposition immediately follows for this case.

Part (i): experts. Suppose $i'(\theta) \geq 0$.

The monotonicity of the stopping profile follows directly from Lemma 2 in the experts environment because the planner can always decide not to use the additional information from the higher type θ in the proof of Lemma 2. Therefore, we can use (48) as the planner's objective.

In the experts environment, $\hat{v}_H(z)$ need not be decreasing and therefore we need to verify that the policy function x^* we get from Proposition 2 is increasing in z and therefore defines a boundary policy. All other parts of the proof of Proposition 2 remain unchanged even when the stopping payoffs are not monotone.

To show the monotonicity of x^*, we redo the part of the proof of Proposition 2 that shows that $g(x, z) > 0$.

$$\begin{aligned}
g(x,z) =& x(1-x)\Big[x\Big(\beta'(z)(\beta(z)-1)\hat{v}'_H(z) - ((\beta(z)-1)\beta''(z) - 2(\beta'(z))^2)\hat{v}_H(z)\Big)\\
&+ (1-x)\Big(\beta'(z)\beta(z)\hat{v}'_L(z) - (\beta(z)\beta''(z) - 2(\beta'(z))^2)\hat{v}_L(z)\Big)\Big]/\\
&\Big[\Big(x(\beta(z)-1)^2\hat{v}_H(z) + (1-x)(\beta(z))^2\hat{v}_L(z)\Big)\beta'(z)\Big]\\
=& x(1-x)\Big[x\Big(\beta'(z)(\beta(z)-1)(v'_H(z) + v_H(z)h^{-1''}(z)(h^{-1'}(z))^{-1})\\
&- ((\beta(z)-1)\beta''(z) - 2(\beta'(z))^2)v_H(z)\Big)\\
&+ (1-x)\Big(\beta'(z)\beta(z)(v'_L(z) + v_L(z)h^{-1''}(z)(h^{-1'}(z))^{-1})\\
&- (\beta(z)\beta''(z) - 2(\beta'(z))^2)v_L(z)\Big)\Big]/\\
&\Big[\Big(x(\beta(z)-1)^2 v_H(z) + (1-x)(\beta(z))^2 v_L(z)\Big)\beta'(z)\Big],
\end{aligned}$$

where we use $v_\omega(z)$ for $v_\omega(h^{-1'}(z))$ and use that $\hat{v}'_\omega(z) = v'_\omega(z)h^{-1'}(z) + v_\omega(z)h^{-1''}(z)$.

The expression is otherwise equivalent to $g(x, q)$ in (9) but with an additional term in the numerator:

$$x(1-x)\beta'(z)h^{-1''}(z)(h^{-1'}(z))^{-1})\Big[x(z)(\beta(z)-1)v_H(z) + (1-x)\beta(z)v_L(z)\Big]. \tag{49}$$

First, notice that $\beta' < 0$, $h^{-1'} < 0$, and $h^{-1''} < 0$ where the last part follows from the assumption that we are in the experts environment. The term inside the brackets is negative for all $x < x^E(z)$ and hence (49) is weakly positive at $x^*(z)$.

Then, $g(x, z) > 0$ follows from the proof of Proposition 2 for the original model because the additional term only makes it larger.

$\square$