

WAR OF ATTRITION WITH AFFILIATED VALUES

Chang Koo Chi¹, Pauli Murto², and Juuso Välimäki*²

¹Department of Economics, Norwegian School of Economics

²Department of Economics, Aalto University School of Business

June 29, 2017

Abstract

We compute symmetric equilibria for the war of attrition when the private information of the two players is binary and affiliated. We cover both the case of common values and affiliated private values. We demonstrate the possibility of non-monotonic symmetric equilibria, i.e. equilibria where the player with a lower type wins with a positive probability. Interpreting players' expenditures as revenue, we compare the expected revenue in the symmetric equilibrium of the war of attrition to related mechanisms, the all-pay auction and standard first- and second-price auctions.

JEL CLASSIFICATION: D44, D82

KEYWORDS: War of Attrition, affiliated signals

1. Introduction

War of attrition, first applied in biology by [Maynard-Smith \[1974\]](#), is a dynamic game where two players compete for an exclusive resource. At each point in time, each player has the option of dropping out from the game or continuing at a cost. The player that drops out last wins the resource at the moment when the other player quits the game.

[Fudenberg and Tirole \[1986\]](#) provide a first analysis of the war of attrition under private information as a Bayesian game with independent private values. [Krishna and Morgan \[1997\]](#) emphasize the interpretation of the Bayesian game as a particular form of an auction with a special payment rule: both bidders pay the bid of the lowest bidder. We follow Krishna and Morgan in using the terminology of bids and bidders in the following. Obviously one should keep in mind also alternative interpretations of the game as an actual contest.

A natural feature in many contest situations is that some common factor affects the players' valuation of the prize or the players' estimates of the value. This observation calls for the analysis

*Chang-Koo Chi: chang-koo.chi@nhh.no; Pauli Murto: pauli.murto@aalto.fi; Juuso Välimäki: juuso.valimaki@aalto.fi

of such models under affiliated values. The paper by [Krishna and Morgan \[1997\]](#) obtains revenue comparisons between the war of attrition and other auction formats in the case where equilibria are monotone. This includes cases with affiliated values, but only under a restricting condition on the model parameters that ensures equilibrium monotonicity.

Our goal in this paper is to examine the war of attrition in the case with affiliated types without any restrictions to monotonic symmetric equilibria. When monotonicity fails, standard methods of analysis are not available. In order to make progress, we assume the simplest possible statistical model for the types of players. Our model is a symmetric model with affiliated binary signals. This includes as special cases the affiliated private values case and the common values case.

We find a complete characterization for the symmetric equilibria of the model. The symmetric equilibrium is always unique and it involves randomizations by both types of players. Given the all-pay nature of the game, it is clear that the symmetric equilibria cannot have atoms. Depending on the parameters of the game, the bid supports of the two types may be non-overlapping, partially overlapping or totally overlapping in the sense that the support of the low type is a subset in the support of the high type. The supports of the individual types are connected and their union is also connected. The lowest bid of zero is always in the support of the low type and the bid support of the high type is unbounded. With totally overlapping supports we have complete rent dissipation since zero is in the support of both types and there are no atoms.

Whenever the bid supports of the two types overlap, there is a positive probability that the low type wins even when the other player is of high type. We emphasize that this type of equilibrium exists even in the case with affiliated private values. This means that the equilibrium allocation in this case is inefficient. In order to compare expected revenues (or expected effort expenditures in the contest interpretation), we need to compare the total surplus and the information rent captured by the high type bidders across different auction formats. Both first-price and second-price auctions are efficient in our setting and result in the same information rent to both types of bidders.¹ We show that the war of attrition always results in a lower information rent to the bidders in symmetric equilibrium than the standard formats. We also show that the efficiency loss relative to these formats is smaller than the reduction in the information rent. As a result, war of attrition outperforms the standard auction formats in terms of expected revenue.

In [Chi, Murto, and Välimäki \[2017\]](#), we compute the allocation and information rent for the closely related model of first price all-pay auction. When comparing war of attrition to this auction format, we find that the revenue ranking depends on the type of model that we have in mind. For common values, war of attrition does better, but with affiliated private values, the all-pay auction sometimes results in higher expected revenues.

¹The equal rent property is a special feature of our binary signal structure.

2. Model

We consider a war of attrition with interdependent valuations. There are two bidders $i, j \in \{1, 2\}$ who compete for a single indivisible object.² Before making a bid, each bidder i privately observes a binary signal $t_i \in \{L, H\}$ with $H > L$. Throughout the paper, we shall refer to a bidder with $t_i = H$ as the high type and a bidder with $t_i = L$ as the low type, respectively. Let $p(t_1, t_2)$ denote the joint probability distribution over signals. We assume p is symmetric over the signals and log-supermodular (log-spm). The symmetric signal structure enables us to define each bidder i 's interim belief on t_j as $p_k(m) \equiv \Pr(t_j = m | t_i = k)$ without the index i . The log-spm of p implies that also the interim belief is log-spm: $p_H(H)p_L(L) \geq p_H(L)p_L(H)$.

After observing her signal, each bidder simultaneously chooses a nonnegative bid, and the highest bidder wins the auction. Both bidders pay the bid of the losing bidder. For each realization of (t_1, t_2) , bidder 1's valuation for the object is $V_{t_1}(t_2)$ and bidder 2's is $V_{t_2}(t_1)$. That is, like in the case of interim belief, we use the subscript for the bidder's own signal and its argument for the opponent's signal. We assume $V_H(m) \geq V_L(m)$ for each $m = L, H$ and $V_H(L) \geq V_L(H)$, implying that for each bidder i , a higher t_i is good news regardless of t_j and t_i has at least as great influence on her own value as t_j . It is worth pointing out two special cases. The model with *common values* satisfies $V_H(L) = V_L(H)$ and the model with *affiliated private values* satisfies $V_H(H) = V_H(L)$ and $V_L(H) = V_L(L)$.

The strategies of the two bidders are given by pairs of distribution functions (F_i^L, F_i^H) , and we denote the supports of the distributions by $\text{supp}(F_i^{t_i})$. For a type vector $t = (t_1, t_2)$, and a bid vector $b = (b_1, b_2)$, we can write the payoff in the war of attrition as follows:

$$u_i(b, t) = \begin{cases} V_{t_i}(t_k) - b_j & \text{if } b_i > b_j, \\ -b_i & \text{if } b_i < b_j, \\ \frac{1}{2}V_{t_i}(t_k) - b_i & \text{if } b_i = b_j, \end{cases}$$

where we assume that each bidder gets the good with probability $\frac{1}{2}$ if the two bids are tied.

A Bayes-Nash equilibrium of the game is a pair of strategies $((F_1^L, F_1^H), (F_2^L, F_2^H))$ such that for all i and all t_i and all $b \in \text{supp}(F_i^{t_i})$,

$$b \in \underset{b'}{\text{argmax}} \sum_{t_k} p_{t_i}(t_k) \left\{ \int_0^{b'} [V_{t_i}(t_k) - s] dF_j^{t_k}(s) - b'(1 - F_j^{t_k}(b')) \right\}.$$

A symmetric Bayes-Nash equilibrium (F_*^L, F_*^H) is a Bayes-Nash equilibrium where $(F_1^L, F_1^H) = (F_2^L, F_2^H) = (F_*^L, F_*^H)$.

²With more than two players, the standard war of attrition where the players observe each other's actions during the game has no symmetric equilibrium. See [Klemperer and Bulow \[1999\]](#) for the analysis of an N-player war of attrition with an appropriate modification to guarantee existence. [Krishna and Morgan \[1997\]](#) analyse a "silent" war of attrition, where the players do not observe each other's actions during the game.

We can rule out atoms in symmetric equilibria since the value of the object is assumed to be strictly positive for all signal realizations. If there was a positive probability of a tie, an arbitrarily small deviation to a higher bid would result in a discrete increase in the winning probability. We can also rule out gaps in the union of the bidding supports. Since the bidders must pay their own bids when losing, a player that bids the lowest bid above such a gap could decrease her payment by deviating downwards without affecting her winning probability. Therefore, a symmetric equilibrium must consist of atomless bid distributions for the two types, where the union of the supports must be a connected set that includes zero.

3. Analysis

In this section, we characterize the unique symmetric equilibrium of the war of attrition and we provide a necessary and sufficient condition for the equilibrium to be in monotone strategies.

To begin, we denote by $\mathbf{F}_* = (F_*^L, F_*^H)$ the symmetric equilibrium, where for each $t_i = L, H$, $F_*^{t_i}(b)$ indicates the probability of the type- t_i 's bid being b or less. It is useful to define the following variable $\phi_{t_i}(t_j) \equiv V_{t_i}(t_j)p_{t_i}(t_j)$. Bidder i 's expected payoff from bidding b with type t_i in the symmetric equilibrium can then be written as:

$$U(b, t_i | \mathbf{F}_*) = \sum_{t_j=L, H} \phi_{t_i}(t_j) \left[F_*^{t_j}(b) + \frac{1}{V_{t_i}(t_j)} \int_0^b F_*^{t_j}(x) dx \right] - b, \quad (1)$$

where we used integration by parts to obtain the expression.

Even though the formula for $U(b, t_i | \mathbf{F}_*)$ looks complicated, a simple perturbation argument of changing the bid of agent i with signal t_i from b to $b + \Delta$ in the interior of the support yields a much simpler condition. This change in the bid is payoff relevant only if $b_j \geq b$. Conditional on $b_j \geq b$, bidder j 's bid is in the interval $[b, b + \Delta]$ if j has observed signal t_j with probability

$$\frac{p_{t_i}(t_j) f_*^{t_j}(b) \Delta}{1 - p_{t_i}(L) F_*^L(b) - p_{t_i}(H) F_*^H(b)}.$$

The payoff from winning at $b + \Delta$ is $V_{t_i}(t_j)$ if j has signal t_j . Hence, if b is in the support of type $t_i = k$ but not of type $t_i = m \neq k$, the following necessary condition for the equalization of the cost and the benefit must hold for type $t_i = k$:³

$$\frac{\phi_k(k) f_*^k(b)}{1 - p_L(L) F_*^L(b) - p_L(H) F_*^H(b)} = 1. \quad (2)$$

Note that since b is not in the support of type m , $F_*^m(b)$ is constant at b , and (2) is a linear first-order differential equation for $F_*^k(b)$.

When b is in the support of both types $t_i = L$ and $t_i = H$, the corresponding necessary

³Notice that whenever $b_j \in [b, b + \Delta]$, the true cost b_j is within Δ from $b + \Delta$ and hence the added cost is approximated by Δ .

conditions must hold for both types:

$$\frac{\phi_L(L) f_*^L(b) + \phi_L(H) f_*^H(b)}{1 - p_L(L) F_*^L(b) - p_L(H) F_*^H(b)} = 1,$$

$$\frac{\phi_H(L) f_*^L(b) + \phi_H(H) f_*^H(b)}{1 - p_H(L) F_*^L(b) - p_H(H) F_*^H(b)} = 1.$$

By rearranging, the conditions can be expressed in a matrix form as follows:

$$\begin{bmatrix} f_*^L(b) \\ f_*^H(b) \end{bmatrix} = \frac{1}{\phi_L(L)\phi_H(H) - \phi_L(H)\phi_H(L)} \begin{bmatrix} \phi_H(H) & -\phi_L(H) \\ -\phi_H(L) & \phi_L(L) \end{bmatrix} \times \begin{bmatrix} 1 - p_L(L) F_*^L(b) - p_L(H) F_*^H(b) \\ 1 - p_H(L) F_*^L(b) - p_H(H) F_*^H(b) \end{bmatrix} \quad (\text{DE})$$

which is a linear differential equation system for $(F_*^L(b), F_*^H(b))$.

Using (2) and (DE) we can construct the symmetric equilibrium of the game. We start from $b = 0$ using initial conditions $F_*^L(0) = F_*^H(0) = 0$, and derive bidding distributions using either (2) or (DE) depending on whether or not (DE) yields a solution with $(f_*^L(b), f_*^H(b)) \gg 0$. It turns out that depending on the parameters, there are three possible configurations for the bid supports, as summarized in Proposition 1 below. The details of the construction as well as the closed form solutions for the bid distributions are given in the proof, which is relegated to Appendix A. It is useful to define the following piece of notation: $\lambda := \frac{p_H(H)}{p_L(H)} > 1$.

Proposition 1. *The war of attrition has a unique symmetric equilibrium \mathbf{F}_* with the following properties:*

(i) *If $\phi_H(L) \geq \phi_L(L) \cdot \lambda$, then the equilibrium is in monotone strategies:*

$$F_*^L(b) = \frac{1}{p_L(L)} \left[1 - \exp\left(-\frac{b}{V_L(L)}\right) \right] \text{ on } \text{supp}[F_*^L] = [0, \bar{B}_L]$$

$$F_*^H(b) = 1 - \exp\left[-\frac{b - \bar{B}_L}{V_H(H)}\right] \text{ on } \text{supp}[F_*^H] = [\underline{B}_H, \infty)$$

where $\bar{B}_L = \underline{B}_H = V_L(L) \log[p_L(H)]$.

(ii) *If $\phi_H(L) \in (\phi_L(L), \phi_L(L) \cdot \lambda)$, then $\text{supp}[F_*^L] = [0, \bar{B}_L]$ and $\text{supp}[F_*^H] = [\underline{B}_H, \infty)$ with $\bar{B}_L > \underline{B}_H > 0$. That is, the two supports are partially overlapping.*

(iii) *If $\phi_H(L) \leq \phi_L(L)$, then $\text{supp}[F_*^L] \subset \text{supp}[F_*^H]$ and both supports begin at zero.*

Proof. See Appendix A. □

The symmetric equilibrium is monotonic under the condition of case (i). By this we mean that a high-type bidder, whenever present, wins against a low type with probability one. If this condition is not satisfied, the equilibrium is non-monotonic in the sense that the object is allocated

with a positive probability to a low type bidder even when the other bidder has a high type. This is not possible in the symmetric equilibrium of the model with independent types.⁴ We relegate to Appendix A the exact functional form of non-monotonic equilibria.

It can be shown that under the condition of case (i) the war of attrition has the following single-crossing property: if an increment of b is desirable to a bidder when her opponent employs a monotone strategy, then the bid increment remains desirable to her at a higher type. This leads to the equilibrium where the two supports are disjoint.

In Chi et al. [2017], we analyze the closely related model of (first-price) all-pay auction and show that the condition for the existence of a monotone strategy equilibrium in that model is $\phi_H(L) \geq \phi_L(L)$. Hence, we see that the war of attrition calls for a stronger condition for the existence of a monotone strategy equilibrium. The equilibrium characterization of the two auction formats also differs in another way. In the all-pay auction there are only two different configurations: either the equilibrium is monotonic (the case corresponding to (i) here) or the supports are fully nested (the case corresponding to (iii) here). The case (ii), where the two supports are partially overlapping is hence unique to the war of attrition.

To better understand the difference between war of attrition and all-pay auction, let us discuss the incentives for deviations of the bidders. Start with the all-pay auction. This is an easier form to analyze, because the bidders always pay their own bid irrespective of the other player's bid. We show in Chi et al. [2017] that also in that format, whenever a monotonic equilibrium exists, the low type bidders earn no expected rents and the bidding supports are connected intervals. Consider the highest bid in the support of the low type bid distribution. By bidding there, the low type bidder wins for sure against a low type bidder but loses for sure against a high type bidder. Equating the payoff from this bid and bidding zero shows that the highest bid in the low-type support must be $p_L(L) V_L(L)$. This same bid is also the lowest bid of the high-type bidders. To have a monotonic equilibrium, the high-type bidder must make a non-negative profit at this bid, which is the case if and only if

$$p_H(L) V_H(L) \geq p_L(L) V_L(L),$$

which is indeed the condition $\phi_H(L) \geq \phi_L(L)$.

A key feature making the analysis of the all-pay auction straight-forward in comparison to the war of attrition is the fact that the local incentives for deviations by the high type are constant in the support of the low bids. This follows from the fact that for the low type to be indifferent throughout her bidding support, her winning probability must be equal to her bid, and hence her bidding density is constant. This means that also for the high type the winning probability must change linearly in her bid, making her payoff linear in her bid.

In the war of attrition, the changes in the expected payment are not linear since it is the hazard rate of the opponent's bid rather than its density that must remain constant across the low type

⁴With independent values, we have $p_H(H) = p_L(H)$ and $p_H(L) = p_L(L)$. It follows that $\lambda = 1$ and $\phi_H(L) \geq \phi_L(L)$, and therefore the condition of case (i) holds.

bidding distribution. Equation (2) with $k = L$ implies that for monotonic equilibria,

$$f_*^L(\bar{B}_L) = \frac{1 - p_L(L)}{\phi_L(L)}. \quad (3)$$

Hence to rule out local deviations for the high type bidders, we must have:

$$\phi_H(H)f_*^L(\bar{B}_L) \geq 1 - p_H(L),$$

which, combined with (3) and the fact that $p_m(H) = 1 - p_m(L)$, $m = H, L$, gives the condition of case (i) in Proposition 1:

$$\phi_H(L) \geq \phi_L(L)\lambda.$$

It is the curvature in the high-type bidders deviation payoff that induces the qualitative difference in the equilibrium characterization of the two games. In the all-pay auction, where the deviation payoff for the high-types is linear, the monotone equilibria fail to exist if and only if a bid of zero is in the support of the high type bidders too. In the war of attrition, where deviation payoff is non-linear, there is a range of parameters where the equilibrium is in non-monotonic strategies and yet the high type earns a positive rent since the bidding support of the high-type does not contain zero (case (ii) of Proposition 1).

We conclude this section by computing the expected rents for the two types of bidders in the symmetric equilibrium of the war of attrition. In order to compute $U(b, t_i | \mathbf{F}_*)$ for $b \in \text{supp}[F_*^{t_i}]$, we first note that zero belongs to $\text{supp}[F_*^L]$ implying right away that the expected rent of the low-type bidders is zero. Combining this fact with the expression (1) evaluated for the low-type at $b = \underline{B}_H$ gives

$$U(\underline{B}_H, L | \mathbf{F}_*) = -\underline{B}_H + p_L(L) \left[V_L(L)F_*^L(\underline{B}_H) + \int_0^{\underline{B}_H} F_*^L(x)dx \right] = 0.$$

Solving this for \underline{B}_H and substituting into the expression (1) evaluated for the high-type at $b = \underline{B}_H$ gives us the expected rent of the high-type bidders in the game:

$$\begin{aligned} U(\underline{B}_H, H | \mathbf{F}_*) &= -\underline{B}_H + p_H(L) \left[V_H(L)F_*^L(\underline{B}_L) + \int_0^{\underline{B}_H} F_*^L(x)dx \right] \\ &= F_*^L(\underline{B}_H) \left[\phi_H(L) - \phi_L(L) - (p_L(L) - p_H(L)) \int_0^{\underline{B}_H} F_*^L(x)dx \right], \end{aligned} \quad (4)$$

where the equilibrium bid distribution F_*^L satisfies at the boundary \underline{B}_H

$$F_*^L(\underline{B}_H) \begin{cases} = 0 & \text{if } \phi_H(L) < \phi_L(L) \\ \in (0, 1) & \text{if } \phi_H(L) \in [\phi_L(L), \phi_L(L) \cdot \lambda) \\ = 1 & \text{if } \phi_H(L) \geq \phi_L(L) \cdot \lambda \end{cases}$$

4. Revenue Comparisons

In this subsection, we compare the expected revenue across different auction formats (first- and second-price auction, all-pay auction and the war of attrition) and provide a revenue ranking. In [Chi et al. \[2017\]](#) we show that the first- and second-price auctions are payoff equivalent in the current binary setting, and therefore we use the term standard auction to encompass both of these formats.⁵

For the war of attrition and the all-pay auction, it may be more natural to interpret payments as effort expended when competing for the prize. Hence the comparison of expected payments can be seen as comparisons between expected equilibrium effort. Similarly the revenues in standard auction formats (first- and second-price auctions) can be interpreted as commitments to effort levels conditional on winning the contest.

To compute the expected revenue from an auction, we subtract the bidders' rent from the total surplus. The auctions differ from each other both in the amount of rent appropriated by the bidders and in the total surplus generated, where the latter difference arises from the possibility of an inefficient allocation in the all-pay mechanisms. In the pure common values case where $V_H(L) = V_L(H)$, the allocation is always efficient and only the bidders' rent matters.

We let $U^M(H)$ denote the expected payoff of the high type bidder in the symmetric equilibrium of an auction of type M for $M \in \{W, A, S\}$, where the superscripts stand for war of attrition, all-pay auction and standard auctions. In [Chi et al. \[2017\]](#), we show that the all-pay auction and the standard auctions have a unique symmetric equilibrium and that the low type earns a zero expected rent in all three auction formats. Since low type bidders earn a zero profit in the war of attrition as well, it is sufficient to consider the expected rents of the high types only. The first two lines in equation (5) below summarize our previous result on standard auctions and the all-pay auction. The third line reproduces the result (4) for the high type bidders from the previous section.

$$U^M(H) = \begin{cases} p_H(L)[V_H(L) - V_L(L)] & \text{for } M = S \\ \max\{\phi_H(L) - \phi_L(L), 0\} & \text{for } M = A \\ F_*^L(\underline{B}_H) \left[\phi_H(L) - \phi_L(L) - (p_L(L) - p_H(L)) \int_0^{\underline{B}_H} F_*^L(x) dx \right] & \text{for } M = W. \end{cases} \quad (5)$$

Because $p_L(L) > p_H(L)$ and $F_*^L(\underline{B}_H) \in [0, 1]$, we obtain from (5) the information rent ranking $U^S(H) > U^A(H) \geq U^W(H)$: standard auctions leave the highest rents, and all-pay auctions leave higher rents than the war of attrition to the high type.⁶

Let Π^M denote the expected revenue to the seller and let L^M denote the surplus loss relative to the socially efficient policy in auction format M . Denote by $\mathbb{P}(n)$ the probability that exactly n

⁵We know from the linkage principle ([Milgrom and Weber \[1982\]](#)) that the second-price auction is revenue-superior to the first-price auction when we allow a richer signal space.

⁶The equality $U^A(H) = U^W(H)$ holds only when both are zero.

of the bidders has high type. Finally, let Π^* denote the expected social surplus from the efficient allocation rule.

We start by comparing the war of attrition to standard auctions. When the bidding supports are disjoint, the symmetric equilibrium of the war of attrition achieves the efficient allocation but reduces the information rents to the high type. Hence the war of attrition dominates standard auctions in terms of expected revenue.

When the bidding supports in the war of attrition are (partly) overlapping, positive bidder rents and efficiency loss coexist (except when zero is in the support of both bid distributions). We show in the proof of the next proposition that the sum of these two components is always lower than the bidder rent in standard auctions, and hence the war of attrition dominates standard auctions.

Proposition 2. *The symmetric equilibrium of the war of attrition generates a higher expected revenue than the symmetric equilibrium of a standard auction.*

Proof. When $\phi_H(L) \geq \phi_L(L) \cdot \lambda$, the unique equilibrium of the war of attrition is in monotone strategies, so there is no surplus loss from inefficient allocation: $L^W = 0$. Hence $\Pi^W > \Pi^S$ follows from $U^W(H) < U^S(H)$.

When $\phi_H(L) \leq \phi_L(L)$, zero is in the support of both types in the war of attrition. In this case, the mechanism extracts full surplus but suffers from inefficiency. The social cost of an inefficient allocation is given by $V_H(L) - V_L(H)$ and can occur only when one of the bidders has a high signal and one has a low signal, an event that takes place with probability $\mathbb{P}(1)$. The surplus loss L^W is hence bounded by $\mathbb{P}(1) [V_H(L) - V_L(H)]$, leading to $\Pi^W > \Pi^S$.

In the remaining case, $\phi_H(L) \in (\phi_L(L), \phi_L(L) \cdot \lambda)$, the war of attrition suffers from an inefficient allocation and at the same time leaves rents to the high type bidders. To complete the proof, first note that a surplus loss can occur only if the low type makes a bid above \underline{B}_H . In this case the surplus is reduced by $V_H(L) - V_L(H)$, which is smaller than $V_H(L) - V_L(L)$. This gives us an upper bound for L^W :

$$L^W < \mathbb{P}(1) \left(1 - F_*^L(\underline{B}_H)\right) \left[V_H(L) - V_L(L)\right].$$

Furthermore, we see from (4) that the high type bidders' information rent is bounded from above by

$$U^W(H) < F_*^L(\underline{B}_H) p_H(L) \left[V_H(L) - V_L(L)\right].$$

By summing over the possible realizations of the types, we get a bound for the total information rent:

$$\sum_{n=1}^2 \mathbb{P}(n) \cdot n \cdot U^W(H) < \sum_{n=1}^2 \mathbb{P}(n) \cdot n \cdot F_*^L(\underline{B}_H) p_H(L) \left[V_H(L) - V_L(L)\right]$$

$$= \mathbb{P}(1) F_*^L(\underline{B}_H) [V_H(L) - V_L(L)],$$

where the last equality follows from the fact that $\mathbb{P}(1) = p_H(L) \sum_{n=1}^2 \mathbb{P}(n) \cdot n$.

By summing these two inequalities and using the fact that $F_*^L(\underline{B}_H) \leq 1$, we get:

$$\Pi^W > \Pi^* - \mathbb{P}(1) [V_H(L) - V_L(L)] = \Pi^S.$$

Therefore, the expected revenue in the war of attrition is higher than in standard auctions in all of the cases. \square

Finally, we compare the expected revenue in the all-pay auction and the war of attrition. As a first step, we obtain from $U^W(H) < U^A(H)$ that in the absence of inefficiencies, the war of attrition generates a higher expected revenue to the seller. In particular, this is the case with common values, where allocation is always efficient.

Proposition 3. *In the model with common values (i.e. when $V_H(L) = V_L(H)$), the symmetric equilibrium of the war of attrition generates a higher expected revenue than the symmetric equilibrium of the all-pay auction.*

Proof. In the model with common values, the ex post value of the object is common to every bidder, so that there is no surplus loss from inefficient allocation in any mechanism. Therefore, we only need to compare the amount of rents given up to the high type. As discussed above, when $\phi_H(L) \leq \phi_L(L)$, there is full rent dissipation in both mechanisms, so $\Pi^W = \Pi^A = \Pi^*$. When $\phi_H(L) > \phi_L(L)$, the all-pay auction leaves a higher information rent than the war of attrition, so $\Pi^W > \Pi^A$ follows. \square

When $V_H(L) > V_L(H)$, allocative inefficiency plays a role too. In particular, there is a range of parameters where the war of attrition involves overlap in bidding supports but the symmetric equilibrium of the all-pay auction is efficient. In this range, allocative efficiency favors the all-pay auction. At the same time, bidder rents are higher in all-pay auction, which favors the war of attrition. At one end of the parameter range (where $\phi_H(L) - \phi_L(L)$ is positive but close to zero), rents in both mechanisms vanish, and hence the inefficiency of the war of attrition dominates revenue comparison. At the other end of the parameter range (where $\phi_H(L)$ approaches $\phi_L(L) \cdot \lambda$), the inefficiency of the war of attrition disappears and the bidder rent difference relative to the all-pay auction dominates.

Proposition 4. *If $V_H(L) > V_L(H)$, the revenue ranking between the symmetric equilibria of the war of attrition and the all-pay auction is ambiguous. In particular, the all-pay auction generates a strictly higher expected revenue than the war of attrition when $\phi_H(L) - \phi_L(L) = 0$, whereas the war of attrition generates a strictly higher expected revenue than the all-pay auction when $\phi_H(L) - \phi_L(L) \cdot \lambda = 0$.*

Proof. When $\phi_H(L) = \phi_L(L)$, the allocation is efficient in the all-pay auction but inefficient in the war of attrition. Moreover, in both mechanisms it is weakly optimal for the high type to bid zero, and therefore $U^A(H) = U^W(H)$. $\Pi^A > \Pi^W$ results from the allocative inefficiency in the war of attrition. When $\phi_H(L) - \phi_L(L) \cdot \lambda = 0$, the bidding supports are disjoint and the allocation is efficient in both mechanisms. Since $U^A(H) > U^W(H)$, however, $\Pi^W > \Pi^A$ in this case. \square

5. Conclusion

This paper takes the first steps towards analyzing the war of attrition when the players have affiliated signals. By allowing for non-monotonic equilibria, we find new types of symmetric equilibria. With sufficient correlation in the signals, the bidding strategies of the two types of bidders have partial or complete overlap depending on the parameter values and rents may be dissipated even in the case of affiliated private values.

Our analysis makes heavy use of the binary signal structure that we have assumed. It is a challenging extension of this work to allow for more general signal structures.

A. Proof of Proposition 1

We analyze the system of linear differential equations (DE). For each type $k = L, H$, define $G^k(b) := 1 - F_*^k(b)$ and $g^k(b) := -f_*^k(b)$. Then we can rewrite (DE) as a homogenous system:

$$\begin{bmatrix} g^L(b) \\ g^H(b) \end{bmatrix} = h \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} G^L(b) \\ G^H(b) \end{bmatrix} := A \begin{bmatrix} G^L(b) \\ G^H(b) \end{bmatrix}, \quad (6)$$

where

$$\begin{aligned} h &:= -\left[\phi_L(L)\phi_H(H) - \phi_L(H)\phi_H(L)\right]^{-1} < 0, \\ a &:= \phi_H(H)p_L(L) - \phi_L(H)p_H(L) > 0, \\ b &:= \phi_H(H)p_L(H) - \phi_L(H)p_H(H) = p_L(H)p_H(H) \left[V_H(H) - V_L(H)\right] > 0, \\ c &:= -\phi_H(L)p_L(L) + \phi_L(L)p_H(L) = p_L(L)p_H(L) \left[V_L(L) - V_H(L)\right] < 0, \\ d &:= -\phi_H(L)p_L(H) + \phi_L(L)p_H(H). \end{aligned}$$

The sign of d is undetermined by the primitives of the model, but as will be seen, in all the cases where overlapping supports occur, parameters must be such that $d > 0$. The characterization of equilibrium is a straightforward phase-plane analysis of (2) and (6) in the plane $[G^L(b), G^H(b)] \in [0, 1] \times [0, 1]$.

We start by defining the region in the plane where overlapping bidding supports are feasible, i.e. the region where (6) admits $g^L(b) \leq 0$ and $g^H(b) \leq 0$ (recall that $g^k(b) := -f_*^k(b)$). Since $a > 0, b > 0$ and $h < 0$, we see from (6) that $g^L(b) \leq 0$ holds always. For $g^H(b) \leq 0$, we need $d > 0$ and $cG^L(b) + dG^H(b) \geq 0$, which defines the region \mathcal{D} where overlapping supports are feasible:

$$\mathcal{D} := \left\{ [G^L, G^H] \in [0, 1] \times [0, 1] : \frac{G^H}{G^L} \geq -\frac{c}{d} \right\}.$$

To derive explicit solutions for the bidding distributions, we note that the matrix A has two real eigenvalues

$$\begin{aligned} r_1 &= \frac{h}{2} \left(a + d - \sqrt{(a-d)^2 + 4bc} \right) < 0, \\ r_2 &= \frac{h}{2} \left(a + d + \sqrt{(a-d)^2 + 4bc} \right) < r_1 < 0. \end{aligned}$$

To see that both r_1 and r_2 are real-valued, write

$$\begin{aligned} a - d &= p_L(L)p_H(H) \left\{ V_H(H) - V_L(L) \right\} + p_L(H)p_H(L) \left\{ V_H(L) - V_L(H) \right\} \\ &= p_L(L)p_H(H) \left\{ V_H(H) - V_L(H) \right\} + p_L(L)p_H(H) \left\{ V_L(H) - V_L(L) \right\} \end{aligned}$$

$$\begin{aligned}
& + p_L(H) p_H(L) \{V_H(L) - V_L(L)\} - p_L(H) p_H(L) \{V_L(H) - V_L(L)\} \\
& \geq p_L(L) p_H(H) \{V_H(H) - V_L(H)\} + p_L(H) p_H(L) \{V_H(L) - V_L(L)\},
\end{aligned}$$

where the inequality follows from $V_L(H) \geq V_L(L)$ and log-supermodularity of p . As a result,

$$(a-d)^2 \geq \left[p_L(L) p_H(H) \{V_H(H) - V_L(H)\} + p_L(H) p_H(L) \{V_H(L) - V_L(L)\} \right]^2.$$

Since

$$-4bc = 4p_L(L) p_H(H) p_L(H) p_H(L) \{V_H(H) - V_L(H)\} \{V_H(L) - V_L(L)\},$$

and since $(s+u)^2 \geq 4su$ for every $s, u \in \Re$, we have $(a-d)^2 + 4bc \geq 0$.

The general solution to (6) is then

$$\begin{bmatrix} G^L(b) \\ G^H(b) \end{bmatrix} = c_1 \begin{bmatrix} k_1 \\ 1 \end{bmatrix} e^{r_1 b} + c_2 \begin{bmatrix} k_2 \\ 1 \end{bmatrix} e^{r_2 b}, \quad (7)$$

where c_1 and c_2 are free parameters and k_1 and k_2 are normalized eigenvector components

$$\begin{aligned}
k_1 &= \frac{a-d - \sqrt{(a-d)^2 + 4bc}}{2c} < 0, \\
k_2 &= \frac{a-d + \sqrt{(a-d)^2 + 4bc}}{2c} < k_1 < 0.
\end{aligned}$$

We now analyze the three mutually exclusive cases of Proposition 1:

Case (i): $\phi_H(L) \geq \phi_L(L) \cdot \lambda$

In this case, $c < d \leq 0$, and therefore $\mathcal{D} = \emptyset$ and there cannot be overlap of supports. The equilibrium (F_*^L, F_*^H) given in case (i) of Proposition 1 follows by analyzing the necessary indifference condition (2) separately for types L and H. First, (2) for type $k = L$,

$$\frac{p_L(L) f_*^L(b) V_L(L)}{1 - p_L(L) F_*^L(b)} = 1,$$

with initial condition $F_*^L(0) = 0$ gives the formula for $F_*^L(b)$ and boundary condition $F_*^L(\bar{B}_L) = 1$ gives the formula for \bar{B}_L . Similarly, (2) for type $k = H$ is

$$\frac{p_H(H) f_*^H(b) V_H(H)}{1 - p_L(L) - p_H(H) F_*^H(b)} = 1,$$

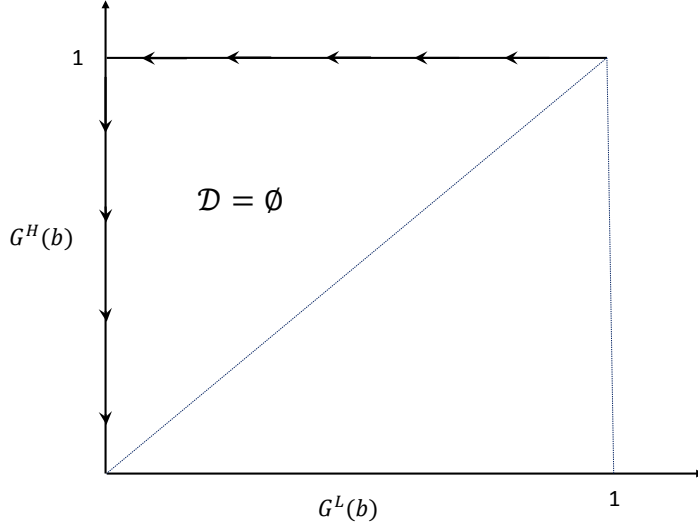


Figure 1: Case (i)

which with initial condition $F_*^H(\bar{B}_L) = 0$ gives the formula for $F_*^H(b)$. See Figure 1, where the equilibrium trajectory is marked by arrows.

Case (ii): $\phi_L(L) \leq \phi_H(L) < \phi_L(L) \cdot \lambda$

In this case, we have $d = -\phi_H(L)p_L(H) + \phi_L(L)p_H(H) > 0$ and

$$\left[G^L(b), G^H(b) \right] \in \mathcal{D} \Leftrightarrow \frac{G^H(b)}{G^L(b)} > -\frac{c}{d} = \frac{\phi_H(L)p_L(L) - \phi_L(L)p_H(L)}{\phi_L(L)p_H(H) - \phi_H(L)p_L(H)} > 1.$$

Since $G^H(0)/G^L(0) = 1 < -c/d$, zero cannot be in the support of type H bid distribution. It is easy to check that the only possibility is that there is a bid interval $[0, \underline{B}_H]$, where only type L is active, and $F_*^L(b)$ in that interval follows from the indifference condition (2) for type L together with initial condition $F_*^L(0) = 0$. The formula for \underline{B}_H is

$$\underline{B}_H = V_L(L) \log \left[\frac{\phi_H(L) - V_L(L)p_H(L)}{\phi_L(L) - V_L(L)p_H(L)} \right] < V_L(L) \log [p_L(H)]^{-1},$$

which follows from the condition

$$\frac{1}{G^L(\underline{B}_H)} = \frac{1}{1 - F_*^L(\underline{B}_H)} = -\frac{c}{d}$$

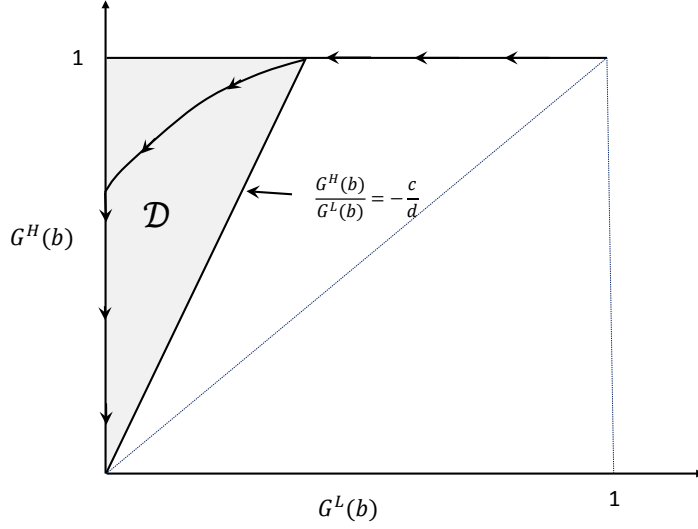


Figure 2: Case (ii)

that defines the point where the equilibrium path hits \mathcal{D} where overlapping supports are feasible (see Figure 2, where \mathcal{D} is the shaded area). It is easy to verify from (6) that we have a trajectory $[G^L(b), G^H(b)]$ with $g^L(b) < g^H(b) < 0$ for $b \in [\underline{B}_H, \bar{B}_L]$, where \bar{B}_L is pinned down by $G^L(\bar{B}_L) = 1$. To get an explicit formula for the bidding distributions, we plug initial conditions $G^L(\underline{B}_H) = -\frac{d}{c}$ and $G^H(\underline{B}_H) = 0$ into (7), which gives:

$$\begin{aligned}\tilde{F}^L(b) &= 1 - G^L(b) = 1 - \tilde{c}_1 \exp[r_1(b - \underline{B}_H)] - \tilde{c}_2 \exp[r_2(b - \underline{B}_H)], \\ \tilde{F}^H(b) &= 1 - G^H(b) = 1 - \tilde{d}_1 \exp[r_1(b - \underline{B}_H)] - \tilde{d}_2 \exp[r_2(b - \underline{B}_H)],\end{aligned}$$

where

$$\begin{aligned}\tilde{c}_1 &= \frac{\left(a + d + \sqrt{(a-d)^2 + 4bc}\right) \left(a - d - \sqrt{(a-d)^2 + 4bc}\right)}{4c\sqrt{(a-d)^2 + 4bc}} < 0, \\ \tilde{c}_2 &= -\frac{\left(a + d - \sqrt{(a-d)^2 + 4bc}\right) \left(a - d + \sqrt{(a-d)^2 + 4bc}\right)}{4c\sqrt{(a-d)^2 + 4bc}} > 0, \\ \tilde{d}_1 &= \frac{\left(a + d + \sqrt{(a-d)^2 + 4bc}\right)}{2\sqrt{(a-d)^2 + 4bc}} > 0,\end{aligned}$$

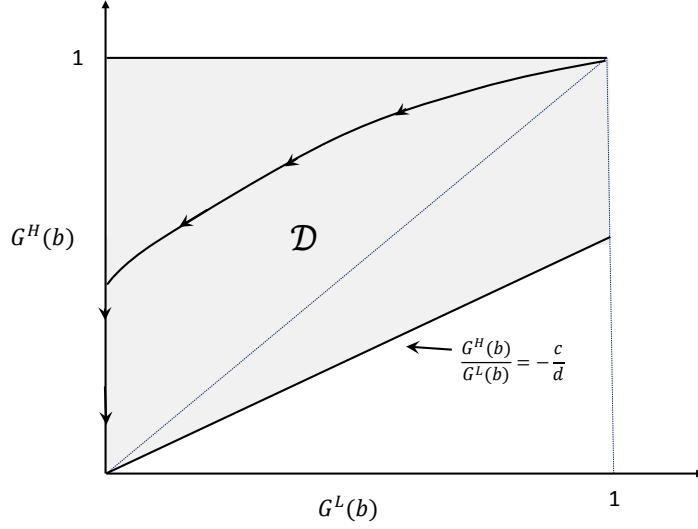


Figure 3: Case (iii)

$$\tilde{d}_2 = -\frac{\left(a + d - \sqrt{(a-d)^2 + 4bc}\right)}{2\sqrt{(a-d)^2 + 4bc}} < 0,$$

and where \bar{B}_L is obtained from boundary condition $\tilde{F}^L(\bar{B}_L) = 1$:

$$\bar{B}_L = \underline{B}_H + \left(-\frac{1}{h}\right) \cdot \frac{\log\left(\frac{d(a-d+\sqrt{(a-d)^2+4bc})-2bc}{d(a-d-\sqrt{(a-d)^2+4bc})-2bc}\right)}{\sqrt{(a-d)^2+4bc}}.$$

The remaining part of F_*^H for $b > \bar{B}_L$ follows directly from indifference condition (2) for type H with initial condition $F_*^H(\bar{B}_L) = \tilde{F}^H(\bar{B}_L)$.

Case (iii): $\phi_H(L) < \phi_L(L)$

In the last case, all points $[G^L(b), G^H(b)]$, $0 \leq G^L(b) \leq G^H(b)$, are in \mathcal{D} , including the initial point $[G^L(0), G^H(0)] = [1, 1]$. System (DE) then pins down a unique trajectory from $[G^L(0), G^H(0)] = [1, 1]$ to $[G^L(\bar{B}_L), G^H(\bar{B}_L)]$, where $G^H(\bar{B}_L) > 0$ and where \bar{B}_L defined by $G^L(\bar{B}_L) = 0$ (see Figure 3). Using initial conditions $G^L(0) = G^H(0) = 1$, we get the formulas for

the bidding distributions from (7):

$$\begin{aligned}\widehat{F}^L(b) &= 1 - \widehat{c}_1 e^{r_1 b} - \widehat{c}_2 e^{r_2 b}, \\ \widehat{F}^H(b) &= 1 - \widehat{d}_1 e^{r_1 b} - \widehat{d}_2 e^{r_2 b},\end{aligned}$$

where

$$\begin{aligned}\widehat{c}_1 &= \frac{\left(a - d - 2c + \sqrt{(a-d)^2 + 4bc}\right) \left(a - d - \sqrt{(a-d)^2 + 4bc}\right)}{4c\sqrt{(a-d)^2 + 4bc}} < 0, \\ \widehat{c}_2 &= \frac{\left(-a + d + 2c + \sqrt{(a-d)^2 + 4bc}\right) \left(a - d + \sqrt{(a-d)^2 + 4bc}\right)}{4c\sqrt{(a-d)^2 + 4bc}} > 0, \\ \widehat{d}_1 &= \frac{a - d - 2c + \sqrt{(a-d)^2 + 4bc}}{2\sqrt{(a-d)^2 + 4bc}} > 0, \\ \widehat{d}_2 &= \frac{-a + d + 2c + \sqrt{(a-d)^2 + 4bc}}{2\sqrt{(a-d)^2 + 4bc}} < 0,\end{aligned}$$

and where \bar{B}_L is determined by $\widehat{F}^L(\bar{B}_L) = 1$:

$$\bar{B}_L = \left(-\frac{1}{h}\right) \cdot \frac{\log\left(\frac{\left(a-d-2c-\sqrt{(a-d)^2+4bc}\right)\left(a-d+\sqrt{(a-d)^2+4bc}\right)}{\left(a-d-2c+\sqrt{(a-d)^2+4bc}\right)\left(a-d-\sqrt{(a-d)^2+4bc}\right)}\right)}{\sqrt{(a-d)^2+4bc}}.$$

Again, the part F_*^H for $b > \bar{B}_L$ follows directly from indifference condition (2) for type 2 with initial condition $F_*^H(\bar{B}_L) = \widehat{F}^H(\bar{B}_L)$.

References

- Chang Koo Chi, Pauli Murto, and Juuso Välimäki. All-pay auctions with affiliated values. Technical report, Aalto University and HECER, 2017.
- Drew Fudenberg and Jean Tirole. A Theory of Exit in Duopoly. *Econometrica*, 54(4):943–960, July 1986. URL <https://ideas.repec.org/a/ecm/emetrp/v54y1986i4p943-60.html>.
- Paul Klemperer and Jeremy Bulow. The Generalized War of Attrition. *American Economic Review*, 89(1):175–189, March 1999. URL <https://ideas.repec.org/a/aea/aecrev/v89y1999i1p175-189.html>.
- Vijay Krishna and John Morgan. An Analysis of the War of Attrition and the All-Pay Auction. *Journal of Economic Theory*, 72(2):343–362, February 1997. URL <http://ideas.repec.org/a/eee/jetheo/v72y1997i2p343-362.html>.
- John Maynard-Smith. The theory of games and the evolution of animal conflicts. *Journal of Theoretical Biology*, 47:209–221, 1974.
- Paul R Milgrom and Robert J Weber. A theory of auctions and competitive bidding. *Econometrica*, 50(5):1089–1122, September 1982. URL <http://ideas.repec.org/a/ecm/emetrp/v50y1982i5p1089-1122.html>.