War of Attrition with Affiliated Values

Chang Koo Chi[1], Pauli Murto[2], and Juuso Välimäki*[2]

[1]Department of Economics, Norwegian School of Economics
[2]Department of Economics, Aalto University School of Business

September 7, 2017

Abstract

We study the war of attrition between two players when the players' signals are binary and affiliated. Our model covers both the case of common values and affiliated private values. We characterize the unique symmetric equilibrium and demonstrate the possibility of non-monotonic symmetric equilibria, i.e. equilibria where the player with a lower signal wins with positive probability. Such an outcome is inefficient in the case of private valuations. We compare the war of attrition to other related mechanisms, the all-pay auction and standard first- and second-price auctions. The war of attrition dissipates the bidders' rents more effectively but at the same time distorts the allocation more severely than the other mechanisms. In terms of expected revenues, the war of attrition dominates the standard auctions, but the ranking against the all-pay auction is ambiguous.

JEL Classification: D44, D82
Keywords: War of Attrition, affiliated signals

1. Introduction

War of attrition, first applied in biology by Maynard-Smith [1974] to study conflict, is a dynamic game where two players compete for an exclusive resource. At any point in time, each player has the option of dropping out from the game or continuing at a cost. The player that drops out last wins the resource at the moment when the other player quits the game.

In economics, the game has been used as a model of exit from a declining industry, R&D races, and political campaigns such as lobbying. Fudenberg and Tirole [1986] provide a first analysis of the war of attrition under private information as a Bayesian game with independent private values. Krishna and Morgan [1997] spin the Bayesian game as a particular form of an auction with a special payment rule: both bidders pay the bid of the lowest bidder. We follow Krishna

*Chang-Koo Chi: chang-koo.chi@nhh.no; Pauli Murto: pauli.murto@aalto.fi; Juuso Välimäki: juuso.valimaki@aalto.fi

and Morgan in using the terminology of bids and bidders in the following. Obviously one should keep in mind the alternative interpretations of the game as an actual contest.

A natural feature in many contest situations is that some common factor affects the players' valuation of the prize or the players' estimates of the value. This observation calls for the analysis of such models under affiliated values. The paper by Krishna and Morgan [1997] analyzes a war of attrition model with interdependent valuations à la Milgrom and Weber [1982] and determines revenue rankings between the war of attrition and other auction forms. Their model includes the case with affiliated values, but relies on a parametric assumption ensuring that the equilibrium bidding strategies are increasing in the bidders' signals. Unlike in standard first- or second-price auctions, such monotonicity is fragile in contests or auctions with the property that even losing bidders forfeit their bids, especially when bidders' types or values are strongly affiliated.[1]

Our goal in this paper is to extend the analysis to cover also the case of equilibria where monotonicity fails. Without monotonicity, the usual methods of analysis for the standard model with a continuum of types also fail.[2] In order to make progress, we assume the simplest possible statistical model for the types of players: each of the two players receives either a high or low signal that is affiliated with her opponent's signal. The model includes as special cases the affiliated private values case and the common values case.

We provide a complete characterization for the symmetric equilibria of this binary model. The symmetric equilibrium is always unique and involves randomizations by both types of players. Given the all-pay nature of the game, it is clear that the symmetric equilibria cannot have atoms. Depending on the correlation between the players' types *and* the effect of the relative value differences, our model gives rise to three different types of symmetric equilibria. If the types are weakly correlated, or if the high-type bidder's valuation is much larger than the low-type bidder's, then the bid supports of the two types are non-overlapping so that the high type bids higher than the low type with probability one. Otherwise, the supports are partially or totally overlapping. When the correlation between signals is high enough or the value differences from the two signals is small, the supports are maximally overlapping in the sense that the support of the low signal bidders is a subset of the support of the high type bidders. In this case, the high signal bidder earns no information rent (since the lowest bid wins with probability zero) and the war of attrition results in full rent dissipation.[3]

Whenever the bid supports of the two types overlap, there is a positive probability that the low type outbids the high type. We emphasize that this type of equilibrium exists even in the

[1] To see the role of affiliation, suppose a bidder's opponent employs an increasing bidding strategy increasing. A bidder with a high type expects a high-type opponent with a higher probability. This increases her expected payment when losing the auction and as a result the best response to an opponent's monotone strategy is not necessarily monotone.

[2] In fact, it is not even clear that a pure strategy equilibrium exists.

[3] In a similar vein, Krishna and Morgan [1997] have shown by one example that the war of attrition can achieve full rent dissipation, and established a link to the first-price all-pay auction: if the all-pay auction extracts all the surplus, then the war of attrition does not have monotone strategies as an equilibrium. In our binary setting, the link between the two all-pay mechanisms is much sharper: the all-pay auction extracts all the surplus *if and only if* the war of attrition does. On the other hand, even if the symmetric equilibrium is non-monotonic in the war of attrition, it is possible that the equilibrium is in monotone strategies in the all-pay auction.

case with affiliated private values, which means that the equilibrium allocation is inefficient.[4] In order to compare expected revenues accruing to the auctioneer (or expected effort expenditures in the contest interpretation) to other auction formats, we need to compare the total surplus and the information rent captured by the high type bidders across different auction formats. The standard auction formats, that is the first-price and second-price auctions, achieve allocative efficiency in our setting and give away the same information rent to the high type.[5] We show that the war of attrition results in a lower information rent to the bidders than the standard formats. We show that the surplus loss from inefficient allocation in the war of attrition is not as large as the reduction in the information rent. As a result, the war of attrition outperforms the standard auction formats in terms of expected revenue.

We also compare the properties of the war of attrition to the closely related contest model of first-price all-pay auction.[6] We have computed the allocation and information rents for that auction format in Chi et al. [2017]. When comparing the war of attrition and the all-pay auction, we find that the revenue ranking depends on the primitives of the model that we have in mind. We show that the information rent extracted by the bidders is always (weakly) higher in the all-pay auction, which favors the war of attrition in terms of expected revenue. This part is fully in line with Krishna and Morgan [1997], who compare rents under the restriction that the equilibria are monotonic in both formats. But when equilibria are non-monotonic, the revenues are also affected by the reduction in the total surplus due to inefficient allocation. It turns out that this loss in surplus is always more severe in the war of attrition, which favors the all-pay auction in terms of seller's revenue. We show that the revenue ranking can go in either direction depending on which of these effects dominates. In the special case of common values, allocation is always efficient and hence the war of attrition performs better.

It is useful to summarize the comparison of the auction formats in our setup by noting that the rankings for rent dissipation and allocative distortions go in the same direction. The war of attrition dissipates more of the bidders' information rents than the all-pay auction, which in turns dissipates more than the standard auctions. Similarly, the war of attrition distorts the allocation more than the all-pay auction, which in turn distorts more than the standard auctions. The rent dissipation effect dominates the distortion effect when contrasting the war of attrition to the standard auctions so that the expected revenue (or effort) is higher in the war of attrition. When contrasting to the all-pay auction, either of the two effects may dominate and hence the revenue ranking is ambiguous.

[4] In Chi, Murto, and Välimäki [2017], we show that the possibility of inefficient allocations may not vanish even in a large (first-price) all-pay auction.

[5] The equal rent property is a special feature of our binary signal structure.

[6] In the first-pay all-pay auction the bidders pay their own bid irrespective of who wins, whereas in the war of attrition the winner pays the second-highest bid. For this reason the war of attrition is sometimes called the second-price all-pay auction. From here on we use the terms all-pay auction and war of attrition to refer to the first- and second-price all-pay auctions, respectively.

2. Binary Model

Two bidders $i, j \in \{1,2\}$ compete for a single indivisible object.[7] At the beginning of the game, each bidder i privately observes a binary signal $t_i \in \{L, H\}$ with $H > L$. Throughout the paper, we shall refer to a bidder with $t_i = H$ as the high type and a bidder with $t_i = L$ as the low type, respectively. Let $p(t_1, t_2)$ denote the joint probability mass function of signals. We assume that p is symmetric over the signals and we define bidder i's interim belief on t_j as $p_k(m) \equiv \Pr(t_j = m | t_i = k)$ without the index i. We also assume that the monotone likelihood ratio property holds: $\frac{p_H(H)}{p_H(L)} \geq \frac{p_L(H)}{p_L(L)}$.

After observing her signal, each bidder simultaneously chooses a non-negative bid, and the highest bidder wins the auction. Both bidders pay the bid of the losing bidder. For each realization (k, m) of (t_1, t_2), bidder 1's valuation for the object is $V_k(m)$ and bidder 2's is $V_m(k)$. That is, like in the case of interim belief, we use the subscript of V for the bidder's own signal and its argument for the opponent's signal. We assume $V_H(m) \geq V_L(m) > 0$ for each $m = L, H$ and $V_H(L) \geq V_L(H)$, implying that for each bidder i, a higher t_i is good news regardless of t_j and t_i has at least as great influence on her own value as t_j. It is worth pointing out two special cases of our model. The model with *common values* satisfies $V_H(L) = V_L(H)$ and the model with *affiliated private values* satisfies $V_H(H) = V_H(L)$ and $V_L(H) = V_L(L)$.

The strategy of bidder i is represented by a pair of distribution functions $\mathbf{F}_i = \left(F_i^L, F_i^H\right)$ of bids, and the support of the bid distribution is denoted by supp $\left(F_i^k\right)$ for each realization k of t_i. Given a type vector $\mathbf{t} = (t_1, t_2)$ and a bid vector $\mathbf{b} = (b_1, b_2)$, we can write bidder i's payoff in the war of attrition as follows:

$$u_i(\mathbf{b}, \mathbf{t}) = \begin{cases} V_{t_i}\left(t_j\right) - b_j & \text{if } b_i > b_j, \\ -b_i & \text{if } b_i < b_j, \\ \frac{1}{2} V_{t_i}\left(t_j\right) - b_i & \text{if } b_i = b_j, \end{cases}$$

where only for ease of notation, we assume that each bidder gets the object with equal probability if the two bids are tied.

A Bayes-Nash equilibrium of the game is a pair of strategies $\langle \mathbf{F}_1, \mathbf{F}_2 \rangle$ such that for each bidder $i \neq j$ and each realization k of t_i, the distribution function F_i^k assigns probability one to the best response to $\mathbf{F}_j$. A symmetric equilibrium $\mathbf{F}_* = \left(F_*^L, F_*^H\right)$ is the one where both bidders adopt the same bidding strategy $\mathbf{F}_1 = \mathbf{F}_2 = \mathbf{F}_*$.

Before turning to the analysis, we make two preliminary observations on the form of symmetric equilibria.[8] First, there cannot be atoms in the equilibrium bid distribution, since the value of the object is assumed to be strictly positive for all signal realizations.[9] If there was a positive prob-

[7] With more than two players, the standard (dynamic) war of attrition where the players observe each other's actions during the game has no symmetric equilibrium. See Klemperer and Bulow [1999] for the analysis of an N-player war of attrition with an appropriate modification that guarantees existence of equilibrium. Krishna and Morgan [1997] analyze a "silent" (static) war of attrition model, where the players do not observe each other's actions during the game.

[8] It is easy to check that these properties also apply to any symmetric equilibrium of (first-price) all-pay auctions.

[9] One implication of this property is that the equilibrium expected payoff must be continuous in bids, which in turn

ability of a tie, an arbitrarily small deviation to a higher bid would result in a discrete increase in the winning probability. Second, the union of the two bid supports, $\text{supp}(F_*^L) \cup \text{supp}(F_*^H)$, must be an interval starting at zero with no internal gaps. Since the bidders must pay their own bids when losing, if there was an internal gap, the bidder who submitted the lowest bid above the gap could decrease her payment by deviating downwards resulting only in a strictly lower payment in the event of losing the auction. Therefore, a symmetric equilibrium of the game must consist of atomless bid distributions for the two types, with the union of the supports being a connected interval that starts from zero.

In light of the first property, we can rule out the case of tie bids in the subsequent equilibrium analysis and define the symmetric equilibrium in a simple way: $\mathbf{F}_*$ is a symmetric equilibrium of the game if for each k, $b \in \text{supp}\left(F_*^k\right)$ implies

$$b \in \underset{b'}{\text{argmax}} \sum_{m=L,H} p_k(m) \left[\int_0^{b'} \left(V_k(m) - x\right) dF_*^m(x) - b'\left(1 - F_*^m\left(b'\right)\right)\right].$$

3. Analysis

In this section, we characterize the unique symmetric equilibrium of the war of attrition and provide a necessary and sufficient condition for the equilibrium to be in monotone strategies.

To begin, we denote by $\mathbf{F}_* = \left(F_*^L, F_*^H\right)$ a symmetric equilibrium where for each $k = L, H$, $F_*^k(b)$ indicates the probability of the type-k's bid being b or less in the equilibrium. Letting $\phi_k(m) \equiv V_k(m)p_k(m)$, we can rewrite the equilibrium expected payoff of bidder i with type $t_i = k$ from bidding b as

$$U(b, k|\mathbf{F}_*) = \sum_{m=L,H} \phi_k(m) \left[F_*^m(b) + \frac{1}{V_k(m)} \int_0^b F_*^m(x) dx\right] - b, \tag{1}$$

where we used integration by parts to obtain the expression.

Even though the payoff formula looks complicated, the following perturbation argument yields a simple equilibrium condition. Consider a change in bid by the bidder i from b to $b + \Delta$ in the interior of $\text{supp}(F_*^k)$. Notice that this bid increment affects bidder i's payoff only when $b_j \geq b$, and that she must be indifferent between those two bids for small enough Δ. Conditional on $t_i = k$, the probability of the event that her opponent who observed type m placed a bid $b_j \in [b, b + \Delta]$ is

$$\frac{p_k(m) f_*^m(b) \Delta}{1 - p_k(L) F_*^L(b) - p_k(H) F_*^H(b)},$$

and the consequential value to bidder i is $V_k(m)$. Hence, if the original bid b is in the support of type k but not of type $m \neq k$ (so the density $f_*^m(b)$ equals zero), the following necessary condition

implies that any two distinctive bids chosen from the support of F_*^k must yield the same expected payoff to the type-k bidder. This indifference condition serves as a key analytical tool in what follows.

for the equalization of the cost and the benefit must hold: [10]

$$\frac{\phi_k(k)f_*^k(b)}{1-p_k(L)F_*^L(b)-p_k(H)F_*^H(b)}=1. \tag{DE-1}$$

Since b is not in the support of type $m\neq k$, the distribution function F_*^m must be constant at b, and thus (DE-1) is a linear first-order differential equation for $F_*^k(b)$.

On the other hand, when the bid b is in the support of both types, the corresponding necessary conditions are

$$\begin{aligned}\frac{\phi_L(L)f_*^L(b)+\phi_L(H)f_*^H(b)}{1-p_L(L)F_*^L(b)-p_L(H)F_*^H(b)} &= 1,\\ \frac{\phi_H(L)f_*^L(b)+\phi_H(H)f_*^H(b)}{1-p_H(L)F_*^L(b)-p_H(H)F_*^H(b)} &= 1.\end{aligned}$$

By rearranging, the conditions can be expressed in a matrix form as follows:

$$\begin{aligned}\begin{bmatrix}f_*^L(b)\\ f_*^H(b)\end{bmatrix}=&\frac{1}{\phi_L(L)\phi_H(H)-\phi_L(H)\phi_H(L)}\begin{bmatrix}\phi_H(H) & -\phi_L(H)\\ -\phi_H(L) & \phi_L(L)\end{bmatrix}\\ &\times\begin{bmatrix}1-p_L(L)F_*^L(b)-p_L(H)F_*^H(b)\\ 1-p_H(L)F_*^L(b)-p_H(H)F_*^H(b)\end{bmatrix},\end{aligned}\tag{DE-2}$$

which is now a system of linear differential equations for $\left(F_*^L(b), F_*^H(b)\right)$.

With (DE-1) and (DE-2) in hand, we can construct the symmetric equilibrium of the game. We start from $b=0$ using the initial condition $F_*^L(0)=F_*^H(0)=0$, and then derive the bid distributions from either (DE-1) or (DE-2), depending on whether or not (DE-2) yields a solution with $\left(f_*^L(b), f_*^H(b)\right)\gg 0$. Figure 1 illustrates the equilibrium construction in the phase plane diagram. It turns out that depending on the parameters, there are three possible cases denoted as Cases (i) - (iii). This corresponds to three possible equilibrium configurations for the bid supports, as we summarize in Proposition 1 below. The details of the construction as well as the closed-form solutions for the bid distributions are given in the proof, which is relegated to Appendix A. To state our principal result, we define $\lambda:=\frac{p_H(H)}{p_L(H)}>1$.

Proposition 1. *The war of attrition has a unique symmetric equilibrium* $\mathbf{F}_*$ *with the following properties:*

(i) If $\phi_H(L)\geq\lambda\phi_L(L)$, then the equilibrium is in monotone strategies:

$$\begin{aligned}F_*^L(b)&=\frac{1}{p_L(L)}\left[1-\exp\left(-\frac{b}{V_L(L)}\right)\right]\quad \textit{on}\quad \operatorname{supp}[F_*^L]=[0,\overline{B}_L]\\ F_*^H(b)&=1-\exp\left[-\frac{b-\overline{B}_L}{V_H(H)}\right]\quad \textit{on}\quad \operatorname{supp}[F_*^H]=[\underline{B}_H,\infty)\end{aligned}$$

[10] Notice that whenever $b_j\in[b,b+\Delta]$, the true cost b_j is within Δ from $b+\Delta$ and hence the added cost is approximated by Δ .

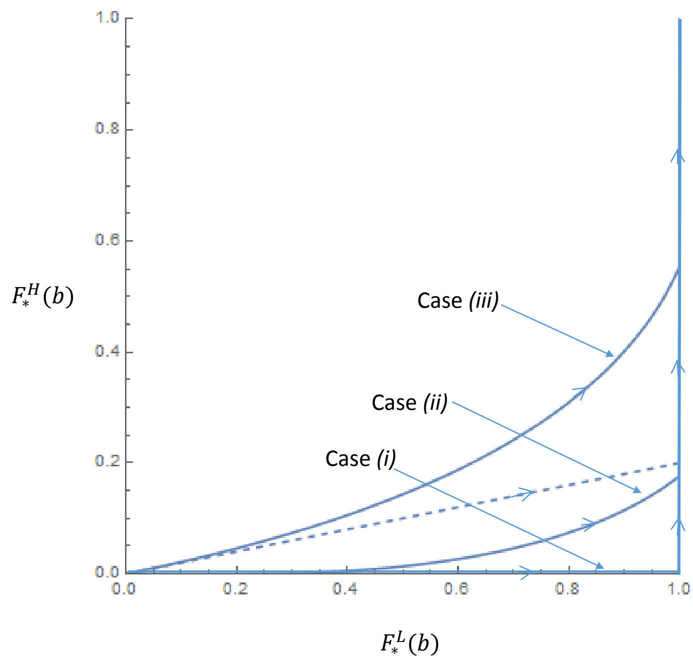


Figure 1: Phase plane diagram of the symmetric equilibrium in the three different cases.

where $\overline{B}_L = \underline{B}_H = -V_L(L)\log[p_L(H)]$.

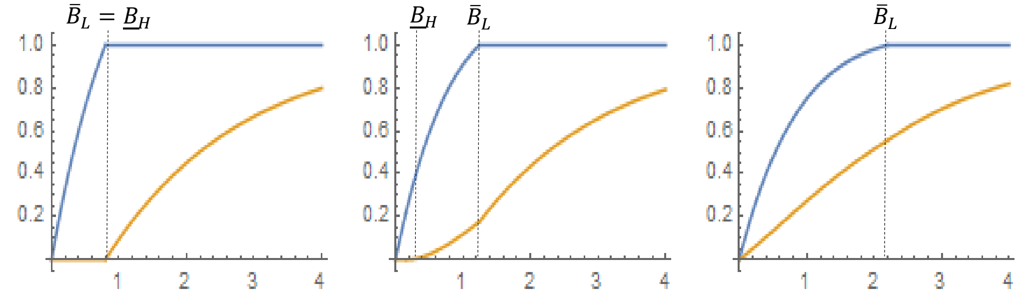
(ii) If $\phi_H(L) \in (\phi_L(L),\ \lambda\phi_L(L))$, *then* $\text{supp}[F_*^L] = [0, \overline{B}_L]$ *and* $\text{supp}[F_*^H] = [\underline{B}_H, \infty)$ *with* $\overline{B}_L > \underline{B}_H > 0$. *That is, the two supports are partially overlapping.*

(iii) If $\phi_H(L) \leq \phi_L(L)$, *then* $\text{supp}[F_*^L] \subset \text{supp}[F_*^H]$ *and both supports begin at zero.*

Proof. See Appendix A. □

Figure 2 illustrates the equilibrium bidding distributions in the three different cases. These figures are drawn in the case of affiliated private values with valuations $V_H(H) = V_H(L) = 2, V_L(H) = V_L(L) = 1$. To get the three different cases, we have varied the correlation in the signals by letting $p_H(H) = p_L(L) = \gamma, p_H(L) = p_L(H) = 1 - \gamma$, where $\gamma = 0.55, 0.65$, and 0.75 in the left, center, and right panels, respectively.

The symmetric equilibrium is monotonic under the condition of case (i). By a monotonic equilibrium we mean that a high-type bidder, whenever present, wins against a low-type bidder with probability one. As the bid distribution of each type has no atoms in any symmetric equilibrium, for every $b^H \in \text{supp}(F_*^H)$ and $b^L \in \text{supp}(F_*^L)$ we must have $b^H \geq b^L$ in a monotonic equilibrium. If the condition $\phi_H(L) \geq \lambda\phi_L(L)$ is not satisfied, the equilibrium is non-monotonic in the sense that the object is allocated with positive probability to a low-type bidder even when he faces a high-type opponent. This inefficient allocation does not arise in the symmetric equilibrium of



(i) *(ii)* *(iii)*

Figure 2: Equilibrium bidding distributions in the three different cases

the model with independent types.[11] We relegate to Appendix A the exact functional form of non-monotonic equilibria.

It can be shown that under the condition of case (i), the war of attrition satisfies the following single-crossing condition (Athey [2001]): if an increment of bid is desirable to the low type when her opponent employs a monotone strategy, then the same bid increment remains desirable to the high type. This leads to the equilibrium where the two supports are disjoint.[12]

In Chi et al. [2017], we analyze the closely related model of (first-price) all-pay auction in the same binary setting, and show that $\phi_H(L) \geq \phi_L(L)$ is the sufficient and necessary condition for its unique symmetric equilibrium to be in monotone strategies. Hence, the result established here suggests that the war of attrition calls for a stronger condition for the existence of a monotone strategy equilibrium. Furthermore, the two auction procedures have another difference in the equilibrium characterization. In the all-pay auction, there are only two different configurations: either the equilibrium is monotonic (the case corresponding to (i) in Proposition 1) or the supports are fully nested (the case corresponding to (iii)). The case (ii), where the two supports are partially overlapping, is hence a distinguished feature of the war of attrition.

To better understand the difference between the two auction formats, let us discuss the incentives for deviations of the high types. Start with the all-pay auction. This is an easier form to analyze, because the bidders always pay their own bid irrespective of the outcome. We show in Chi et al. [2017] that the unique symmetric equilibrium of the all-pay auction, regardless of whether it is monotonic or not, retains at least following two properties: the support of the low type starts at zero and the union of the two bid supports is a connected interval. With this result in hand, suppose both bidders employ a monotone bidding strategy and consider the highest bid

[11]When the bidders' types are independent, we have $p_H(H) = p_L(H)$ and $p_H(L) = p_L(L)$. Hence it follows that $\lambda = 1$ and $\phi_H(L) \geq \phi_L(L)$, and therefore the condition of case (i) is always satisfied.

[12]More precisely, it can be shown that when the condition $\phi_H(L) \geq \lambda\phi_L(L)$ is satisfied, bidder i's expected payoff $U(b, k|\mathbf{F}_j)$ exhibits increasing differences in $(b; k)$: $\partial U(b, L|\mathbf{F}_j)/\partial b < \partial U(b, H|\mathbf{F}_j)/\partial b$. Hence it follows from Milgrom and Shannon [1994] that $\text{supp}(F_i^L) \leq \text{supp}(F_i^H)$ in the strong set order.

in the support of the low-type bid distribution, denoted $\overline{B}_L := \sup \text{supp}(F_*^L)$. By submitting that bid, the low type wins for sure against a low-type opponent but loses for sure against a high-type opponent, so that her payoff is $V_L(L)p_L(L) - \overline{B}_L$. Since the low type must be indifferent between bidding $\overline{B}_L$ and zero, this highest bid must be $\overline{B}_L = V_L(L)p_L(L)$. Furthermore, as the union of the two supports has no internal gap, $\overline{B}_L$ also constitutes the lowest bid of the high-type support. In order to have such monotone strategies as an equilibrium, however, the high-type bidder must make a non-negative profit at $\overline{B}_L$, which is the case if and only if

$$p_H(L)\, V_H(L) - \overline{B}_L \geq 0, \text{ or } \phi_H(L) \geq \phi_L(L).$$

A salient feature making the analysis of the all-pay auction straight-forward in comparison to the war of attrition is the fact that the local incentives for deviations by the high type are constant in the support of the low type. This follows from the fact that for the low type to be indifferent throughout her bidding support, her winning probability must be equal to her bid, and hence her bidding density is constant. This means that also for the high type the winning probability must change linearly in her bid, making her payoff linear in her bid. Consequently, the payoff to deviations by the high type changes linearly and thus her best-response has the bang-bang property. The linear dotted curve in Figure 1 shows the equilibrium path of the all-pay auction in the phase plane with the same parameters as in Case (iii) of the war of attrition. This illustrates why the case of partially overlapping supports does not appear in the all-pay auction (see Chi et al. [2017] for detailed analysis).

In the war of attrition the high type's payoff for deviations does not change linearly in their bids, since it is the hazard rate of the opponent's bid rather than its density function that must remain constant across the support of the low type's bid distribution. To understand the condition for monotonic equilibria, suppose again that both bidders employ a monotone bidding strategy. It follows from the equation (DE-1) with $k = L$ and $b = \overline{B}_L$ that in a monotonic equilibrium (i.e., $F_*^H(\overline{B}_L) = 0$), the low type's density function of bids must satisfy

$$f_*^L(\overline{B}_L) = \frac{1 - p_L(L)}{\phi_L(L)}. \tag{2}$$

At the same time, in order to rule out local deviations from $\overline{B}_L$ by the high type, we must have:

$$\frac{\phi_H(L) f_*^L(\overline{B}_L)}{1 - p_H(L)} \geq 1, \text{ or } f_*^L(\overline{B}_L) \geq \frac{1 - p_H(L)}{\phi_H(L)},$$

which, combined with (2) and the fact that $p_m(H) = 1 - p_m(L)$, $m = H, L$, gives the condition of case (i) in Proposition 1:

$$\phi_H(L) \geq \lambda \phi_L(L).$$

It is the curvature of the deviation payoff to the high type that induces the qualitative difference in the equilibrium characterization between the two games. In the all-pay auction where the deviation payoff for the high types is linear, the monotone equilibria fail to exist if and only if a bid of zero is in the bidding support of the high type. In the war of attrition, on the other hand, the high type's deviation payoff is non-linear. As a result, there is a range of parameters where the equilibrium is in non-monotonic strategies and yet the high type earns a positive rent, since the bidding support of the high type does not contain zero (like case (ii) of Proposition 1).

We conclude this section by computing the expected rents for the two types in the symmetric equilibrium of the war of attrition. For this purpose, we first note that zero belongs to $\mathrm{supp}(F_*^L)$ in all of the cases in Proposition 1, implying right away that the low types earn no rents, which is to be expected. In contrast, the rents for the high type vary in different cases. To handle the three cases at one time, consider the lowest bid in the bid support of the high type, denoted by $\underline{B}_H := \inf \mathrm{supp}(F_*^H)$. Since the union of the two supports has no internal gaps, the low type must be indifferent between bidding zero and $\underline{B}_H$. This observation leads us to an alternative expression of $\underline{B}_H$. Getting the low type's payoff from (1) and then setting $U(\underline{B}_H, L|\mathbf{F}_*) = 0$ gives

$$\underline{B}_H = p_L(L) \left[V_L(L) F_*^L(\underline{B}_H) + \int_0^{\underline{B}_H} F_*^L(x) dx \right]. \tag{3}$$

On the other hand, by submitting the bid $\underline{B}_H$, the high type earns an expected payoff of

$$\begin{aligned} U(\underline{B}_H, H|\mathbf{F}_*) &= -\underline{B}_H + p_H(L) \left[V_H(L) F_*^L(\underline{B}_H) + \int_0^{\underline{B}_H} F_*^L(x) dx \right] \\ &= F_*^L(\underline{B}_H) \Big[\phi_H(L) - \phi_L(L) \Big] - (p_L(L) - p_H(L)) \int_0^{\underline{B}_H} F_*^L(x) dx, \end{aligned} \tag{4}$$

where we used (3) to obtain the second expression. Hence we derived the high types' information rents (4) in terms of $\underline{B}_H$. At this boundary, the low type's equilibrium bid distribution must satisfy

$$F_*^L(\underline{B}_H) \begin{cases} = 0 & \text{if } \phi_H(L) < \phi_L(L) \\ \in (0,1) & \text{if } \phi_H(L) \in [\phi_L(L), \lambda\phi_L(L)) \\ = 1 & \text{if } \phi_H(L) \geq \lambda\phi_L(L). \end{cases}$$

4. Payoff Comparisons

In this subsection, we contrast the rent division and efficiency properties of the war of attrition to other auction formats, the first- and second-price auctions and the all-pay auction. We then provide some revenue ranking results. In Chi et al. [2017], we show that in the current binary setting the first- and second-price auctions are payoff-equivalent, and therefore we shall use the term "standard auction" to encompass both of theses formats.[13]

For the war of attrition and the all-pay auction, it may be more natural to interpret payments

[13] We know from the linkage principle (Milgrom and Weber [1982]) that when we allow a richer signal space, the second-price auction is revenue-superior to the first-price auction.

as effort expended when competing for a prize. Hence the comparison of expected revenues can be seen as comparisons between expected equilibrium effort. Similarly, the revenues in standard auction formats (first- and second-price auctions) can be interpreted as commitments to effort levels conditional on winning the contest.

To compute the expected revenue from an auction, we subtract the bidders' rent from the total surplus. When the symmetric equilibrium is non-monotonic, different auction procedures may lead to not just a different amount of rents appropriated by bidders but a different total surplus generated, where the latter difference arises from the possibility of an inefficient allocation in the all-pay mechanisms. We start the analysis by ranking the auction formats in terms of bidders' information rents and then in terms of allocational efficiency. After that, we provide revenue rankings.

4.1. Bidder Rents

We let $U^M(H)$ denote the expected payoff of the high type in the symmetric equilibrium of an auction of type M, where $M \in \{W, A, S\}$ stands for war of attrition (W), all-pay auction (A) and standard auctions (S). In Chi et al. [2017], we show that the all-pay auction and the standard auctions have a unique symmetric equilibrium in which the low type earns no rents. As even in the war of attrition the rent of the low type is fully dissipated, it is sufficient for revenue comparisons to consider the expected rents of the high type only. The first two lines in equation (5) below summarize our previous result on standard auctions and the all-pay auction. The third line reproduces the result (4) from the previous section.

$$U^M(H) = \begin{cases} p_H(L)\Big[V_H(L) - V_L(L)\Big] & \text{for } M = S \\ \max\Big\{\phi_H(L) - \phi_L(L),\ 0\Big\} & \text{for } M = A \\ F_*^L(\underline{B}_H)\Big[\phi_H(L) - \phi_L(L)\Big] - (p_L(L) - p_H(L))\int_0^{\underline{B}_H} F_*^L(x)dx & \text{for } M = W. \end{cases} \tag{5}$$

This expression gives us the ranking of the auction formats in terms of bidder rents:

Proposition 2. *The standard auctions leave a strictly higher rent to the bidders than the other formats, and the all-pay auction leaves a higher rent than the war of attrition. This holds strictly whenever the rent in the all-pay auction is non-zero.*

Proof. Because $p_L(L) > p_H(L)$ and $F_*^L(\underline{B}_H) \in [0,1]$, we obtain from (5) the ranking of the high type bidder's payoff across the auction formats: $U^S(H) > U^A(H) \geq U^W(H)$, where the equality $U^A(H) = U^W(H)$ holds only when $U^A(H) = U^W(H) = 0$. Since the low type bidder gets zero in all the formats, the result follows. □

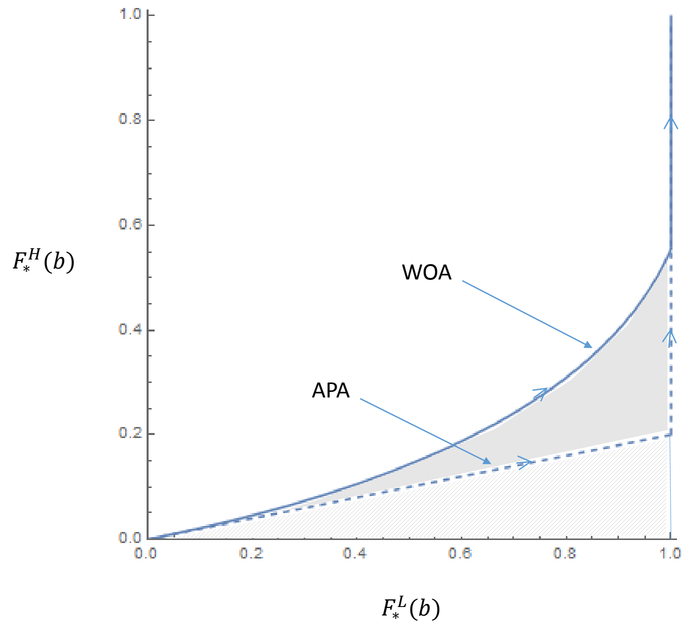


Figure 3: Probability of misallocation: war of attrition vs. all-pay auction

4.2. Allocational Efficiency

A misallocation takes place when a low-type bidder wins the auction and a high-type bidder is present.[14] This is the only source of inefficiency in the model. We have shown in Chi et al. [2017] that a misallocation never takes place in the symmetric equilibrium of the standard auctions. We now compare the probability of misallocation between the war of attrition and the all-pay auction.

To compute the probability of misallocation, it is useful to investigate the equilibrium path in the $\left(F_*^L(b), F_*^H(b)\right)$ phase plane as $b \in [0, \infty)$. The curve marked as WOA in Figure 3 shows the equilibrium path of the war of attrition in the case (iii) of Proposition 1 (the parameters are the same as used in Figures 1 and 2). As b increases, both $F_*^H(b)$ and $F_*^L(b)$ increase in such a way that the path bends upwards until b reaches $\overline{B}_L$, where $F_*^L\left(\overline{B}_L\right) = 1$. Figure 3 shows also the corresponding path in the all-pay auction with the same parameter values, marked as APA. This curve starts at the same slope in the origin, but in contrast to the war of attrition it moves linearly at a constant slope. We show in the proof of Proposition 3 below that this distinction between the two paths is a general feature in the case where the rents are fully dissipated in both auctions.

Let us now consider the probability of misallocation. Conditional on one bidder having a low signal and one having a high signal, we are interested in the probability of the low type winning

[14]We use the term "misallocation" across all the cases even if the allocation has no efficiency consequences in the special case of common values.

the auction, which can be written as:

$$\int_0^{\overline{B}_L} F_*^H(b)\,\frac{dF_*^L(b)}{db}db.$$

By changing the variable of integration to $\beta := F_*^L(b)$, we can write this integral as

$$\int_0^1 F_*^H(\beta)\,d\beta,$$

which is just the area under the equilibrium path illustrated by shaded regions in Figure 3. The probability of misallocation in the case of the all-pay auction is given by the light shaded area while in the war of attrition it is given by the sum of the two shaded areas. We can hence conclude that the probability of misallocation is higher in the case of the war of attrition and we have an unambiguous ranking of the auction formats in terms of their efficiency:

Proposition 3. *The probability of misallocation is higher in the war of attrition than in the all-pay auction. This holds strictly whenever the probability of misallocation is non-zero in the war of attrition. The probability of misallocation is zero in the standard auctions.*

Proof. We compare the probability of misallocation between the war of attrition and the all-pay auction separately in the three cases of Proposition 1.

In Case (i) both formats have a monotonic equilibrium so that the probability of misallocation is zero.

In Case (ii) the equilibrium in the war of attrition features some overlap between the bidding distributions. Therefore the probability of misallocation is strictly positive. In contrast, as we have shown in Chi et al. [2017], the equilibrium in the all-pay auction is monotonic and the probability of misallocation is zero.

In Case (iii), there is overlap in the bidding supports in both auction formats. We computed in the proof of Proposition 1 the equilibrium bidding distributions as:

$$\begin{aligned}\widehat{F}^L(b) &= 1-\widehat{c}_1e^{r_1b}-\widehat{c}_2e^{r_2b},\\ \widehat{F}^H(b) &= 1-\widehat{d}_1e^{r_1b}-\widehat{d}_2e^{r_2b},\end{aligned}$$

where r_1, r_2, $\widehat{c}_1$, $\widehat{c}_2$, $\widehat{d}_1$, and $\widehat{d}_2$ are real numbers such that

$$r_2 < r_1 < 0,$$
$$\widehat{c}_1 < 0, \quad \widehat{c}_2 > 0, \quad \widehat{d}_1 > 0, \quad \widehat{d}_2 < 0.$$

The slope of the equilibrium path can then be written as

$$\frac{d\widehat{F}_*^H(b)}{d\widehat{F}_*^L(b)} = \frac{-\widehat{d}_1 r_1 e^{r_1 b} - \widehat{d}_2 r_2 e^{r_2 b}}{-\widehat{c}_1 r_1 e^{r_1 b} - \widehat{c}_2 r_2 e^{r_2 b}} = \frac{\widehat{d}_1 + \widehat{d}_2 \frac{r_2}{r_1} e^{(r_2 - r_1)b}}{\widehat{c}_1 + \widehat{c}_2 \frac{r_2}{r_1} e^{(r_2 - r_1)b}},$$

where the last step divides both sides by a positive number $-r_1 e^{r_1 b}$ so that both the nominator and denominator remain positive. Given that $\widehat{d}_2 < 0$ and $\widehat{c}_2 > 0$, we see that the nominator is increasing in b while the denominator is decreasing in b, so that the slope $\frac{d\widehat{F}_*^H(b)}{d\widehat{F}_*^L(b)}$ increases in b. In other words, the equilibrium path bends upwards as b increases, as displayed in Figure 3.

We have shown in Chi et al. [2017] that the corresponding bidding distributions in the two-player all-pay auction are uniform. Denoting them by $\widehat{G}_*^L(b)$ and $\widehat{G}_*^H(b)$, we can write them in terms of the notation used here as

$$\begin{aligned} \widehat{G}_*^L(b) &= -\left(r_1\widehat{c}_1 + r_2\widehat{c}_2\right) b \\ \widehat{G}_*^H(b) &= -\left(r_1\widehat{d}_1 + r_2\widehat{d}_2\right) b, \end{aligned}$$

so it follows that

$$\frac{\partial \widehat{G}_*^H(b)}{\partial \widehat{G}_*^L(b)} = \frac{\widehat{d}_1 r_1 + \widehat{d}_2 r_2}{\widehat{c}_1 r_1 + \widehat{c}_2 r_2} = \lim_{b \downarrow 0} \frac{d\widehat{F}_*^H(b)}{d\widehat{F}_*^L(b)}.$$

Hence, the slopes of the equilibrium paths in the two auction formats coincide at the origin. Therefore the area under the equilibrium path of the war of attrition is strictly larger than that of the all-pay auction, which means that the probability of misallocation is higher in the war of attrition.

We have shown in Chi et al. [2017] that both standard auction formats have a unique symmetric equilibrium in which the probability of misallocation is zero.

□

4.3. Revenue Rankings

For each auction procedure M, let Π^M denote the expected revenue accruing to the seller and let L^M denote the surplus loss relative to the socially efficient policy. Also, for each $n = 0, 1, 2$, let $\mathbb{P}(n)$ denote the probability of the event that exactly n of the two bidders receives the signal H. Finally, let $\Pi^\star$ denote the expected social surplus from the efficient allocation rule. With these symbols, we can write the expected revenue in auction M as

$$\Pi^M = \Pi^\star - L^M - \sum_{n=1}^{2} \mathbb{P}(n) \cdot n \cdot U^M(H).$$

We start by comparing the war of attrition to standard auctions. When the bidding supports are disjoint in the symmetric equilibrium, the war of attrition achieves the efficient allocation but, compared to standard auctions, reduces the information rents given away to the high type, leading to a higher expected revenue. In contrast, when the bidding supports are (partly) overlapped, the revenue loss due to rents for the high types and the efficiency loss coexist (except when zero is in the support of both bid distributions). We show in the proof of the next proposition that the sum of these two components is always lower than the revenue loss from rents in standard auctions, and hence the war of attrition outperforms standard auctions in either case.

Proposition 4. *The symmetric equilibrium of the war of attrition generates a higher expected revenue than the symmetric equilibrium of a standard auction.*

Proof. When $\phi_H(L) \geq \phi_L(L) \cdot \lambda$, the unique equilibrium of the war of attrition is in monotone strategies, so there is no surplus loss from inefficient allocation: $L^W = 0$. Hence $\Pi^W > \Pi^S$ is immediate from $U^W(H) < U^S(H)$.

When $\phi_H(L) \leq \phi_L(L)$, zero is in the support of both types in the war of attrition. In this case, the mechanism extracts full surplus but suffers from inefficiency. The social cost of the inefficient allocation amounts to $V_H(L) - V_L(H)$ and can occur only when one of the bidders is of the high type and the other is of the low type. This event takes place with probability $\mathbb{P}(1)$. The surplus loss L^W is hence bounded by $\mathbb{P}(1)\,[V_H(L) - V_L(H)]$, which is less than $\mathbb{P}(1)\,[V_H(L) - V_L(L)]$, i.e. the expected information rent given to the bidders in the standard auction formats. Therefore, $\Pi^W > \Pi^S$.

In the remaining case, $\phi_H(L) \in (\phi_L(L), \phi_L(L) \cdot \lambda)$, the war of attrition suffers from an inefficient allocation and at the same time leaves rents to the high type. To complete the proof, first note that a surplus loss can occur only if the low type makes a bid above $\underline{B}_H$. In this case the surplus is reduced by $V_H(L) - V_L(H)$, which is smaller than $V_H(L) - V_L(L)$. This gives us an upper bound for L^W:

$$L^W < \mathbb{P}(1)\left(1 - F_*^L\left(\underline{B}_H\right)\right)\left[V_H(L) - V_L(L)\right].$$

Furthermore, we see from (4) that the high-type bidders' information rent is bounded from above by

$$U^W(H) < F_*^L\left(\underline{B}_H\right) p_H(L)\left[V_H(L) - V_L(L)\right].$$

By summing over the possible realizations of the types, we get a bound for the total information rent:

$$\begin{aligned}\sum_{n=1}^{2} \mathbb{P}(n) \cdot n \cdot U^W(H) &< \sum_{n=1}^{2} \mathbb{P}(n) \cdot n \cdot F_*^L\left(\underline{B}_H\right) p_H(L)\left[V_H(L) - V_L(L)\right] \\ &= \mathbb{P}(1) F_*^L\left(\underline{B}_H\right)\left[V_H(L) - V_L(L)\right],\end{aligned}$$

where the last equality follows from the fact that $\mathbb{P}(1) = p_H(L) \sum_{n=1}^{2} \mathbb{P}(n) \cdot n$. Combining the two upper bounds and using the fact that $F_*^L\left(\underline{B}_H\right) \leq 1$, we obtain:

$$\Pi^W > \Pi^\star - \mathbb{P}(1)\left[V_H\left(L\right) - V_L\left(L\right)\right] = \Pi^S.$$

Therefore, the expected revenue in the war of attrition is higher than in standard auctions in all of the cases. □

In the same manner, we can determine the revenue ranking between all-pay and standard auctions. When the equilibrium is monotonic, the all-pay auction achieves allocative efficiency and gives away lower rents to the high type than standard auctions. When the equilibrium is non-monotonic, the all-pay auction suffers from the allocative inefficiency that curtails revenues, but the auction fully extracts the information rents from both types. We show in Chi et al. [2017] that the revenue loss arising from inefficiency is lower than the loss from rents in standard auctions, and therefore the all-pay auction is superior in revenues to standard auctions in any cases.

We turn next to the comparison between the two all-pay mechanisms. As a first observation we note that the rent ranking $U^W(H) \leq U^A(H)$ implies that in the absence of allocative inefficiencies, the war of attrition generates a higher expected revenue to the seller. In particular, this is the case with common values, where allocation is always efficient.

Proposition 5. *In the model with common values (i.e. when $V_H(L) = V_L(H)$), the symmetric equilibrium of the war of attrition generates a higher expected revenue than the symmetric equilibrium of the all-pay auction.*

Proof. In the model with common values, the ex post value of the object is common to every bidder, so that there is no surplus loss from inefficient allocation in any mechanisms. Therefore, we only need to compare the amount of rents given up to the high type. By Proposition 2, all-pay auction leaves a higher information rent than the war of attrition, so $\Pi^W \geq \Pi^A$ follows. □

When $V_H(L) > V_L(H)$, we have to take into account the surplus loss from allocative inefficiency as well as information rents. If $\phi_H(L) - \phi_L(L) \leq 0$, then both auction formats dissipate all the bidder rents, so that the more severe inefficiency of the war of attrition implies higher revenues for the all-pay auction. If $\phi_H(L) - \lambda\phi_L(L) \geq 0$, then both auction formats have a monotonic equilibrium with no allocational inefficiencies, and hence the more effective rent dissipation by the war of attrition generates higher revenue. Note that this latter finding is in full accord with the revenue ranking results in Krishna and Morgan [1997].

There is also a range of parameters $\phi_H(L) \in [\phi_L(L), \lambda\phi_L(L))$ where the war of attrition involves overlap in bidding supports but the symmetric equilibrium of the all-pay auction is efficient. In this range, allocative efficiency favors the all-pay auction, whereas the rent for the high type favors the war of attrition. At one end of the parameter range (where $\phi_H(L) - \phi_L(L)$ is positive but close to zero), rents in both mechanisms vanish, and hence the inefficiency of the war of attrition dominates revenue comparison. At the other end of the parameter range (where $\phi_H(L)$ approaches $\lambda\phi_L(L)$ from the left), the inefficiency of the war of attrition disappears and the bidder rent difference relative to the all-pay auction dominates.

Proposition 6. *If $V_H(L) > V_L(H)$, the revenue ranking between the symmetric equilibria of the war of attrition and the all-pay auction is ambiguous. In particular, the all-pay auction generates a strictly higher expected revenue than the war of attrition when $\phi_H(L) - \phi_L(L) \leq 0$, whereas the war of attrition generates a strictly higher expected revenue than the all pay auction when $\phi_H(L) - \lambda\phi_L(L) \geq 0$.*

Proof. When $\phi_H(L) \leq \phi_L(L)$, zero is in the support of both types in both auction formats, and hence the rents are fully dissipated; $U^A(H) = U^W(H) = 0$. By Proposition 3, the probability of misallocation is strictly higher in the war of attrition than in the all-pay auction. With $V_H(L) > V_L(H)$ this implies $L^W > L^A$ and so $\Pi^A > \Pi^W$.

When $\phi_H(L) - \lambda\phi_L(L) \geq 0$, the bidding supports are disjoint and the allocation is efficient in both mechanisms and $L^W = L^A = 0$. By Proposition 2 we have $U^A(H) > U^W(H)$, and therefore $\Pi^A < \Pi^W$. □

5. Conclusion

This paper takes the first steps towards analyzing the war of attrition when the players have affiliated signals. Taking advantage of simple binary signal structures, we provided a complete characterization of the unique symmetric equilibrium and identified tight conditions under which the equilibrium is in monotone or non-monotone strategies. The condition for monotonicity is not satisfied if the types are strongly affiliated or if the high type is not sufficiently good news for the bidder's valuation. In either case, the bidding supports of the two types of bidders have a partial or complete overlap even in the case of affiliated private values, thereby leading to an inefficient allocation of the object.

Comparing the war of attrition to the all-pay auction in the same binary setting, we found that the same parameter condition applies to full rent dissipation in both auction formats. However, there are notable difference in terms of rent division and efficiency properties. We show that the war of attrition dissipates the bidder information rents more effectively than the all-pay auction but at the same time distorts allocation more. The standard first- and second-price auctions feature less rent dissipation and allocational distortions than either of the all-pay formats.

Our analysis makes heavy use of the binary signal structure that we have assumed. The structure enables us to derive the equilibrium condition, the system of differential equations, in a simple and tractable way even when monotonicity fails. It still remains an open question how to extend the analysis presented here to a general signal structure.

A. Proof of Proposition 1

We analyze the system of linear differential equations (DE-2). For each type $k = L, H$, define $G^k(b) := 1 - F_*^k(b)$ and $g^k(b) := -f_*^k(b)$. Then we can rewrite (DE-2) as a homogeneous system:

$$\begin{bmatrix} g^L(b) \\ g^H(b) \end{bmatrix} = h \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} G^L(b) \\ G^H(b) \end{bmatrix} := A \begin{bmatrix} G^L(b) \\ G^H(b) \end{bmatrix}, \tag{6}$$

where

$$\begin{aligned}
h &:= -\left[\phi_L(L)\phi_H(H) - \phi_L(H)\phi_H(L)\right]^{-1} < 0, \\
a_1 &:= \phi_H(H)p_L(L) - \phi_L(H)p_H(L) > 0, \\
a_2 &:= \phi_H(H)p_L(H) - \phi_L(H)p_H(H) = p_L(H)\, p_H(H)\left[V_H(H) - V_L(H)\right] > 0, \\
a_3 &:= -\phi_H(L)p_L(L) + \phi_L(L)p_H(L) = p_L(L)\, p_H(L)\left[V_L(L) - V_H(L)\right] < 0, \\
a_4 &:= -\phi_H(L)p_L(H) + \phi_L(L)p_H(H).
\end{aligned}$$

Unlike the sign of the other parameters, the sign of a_4 is undetermined by the primitives of the model, but as will be seen, whenever the two bid supports are overlapped in equilibrium, we have $a_4 > 0$. The desired symmetric equilibrium can be characterized by a straightforward phase-plane analysis of (DE-1) and (6) over the plane $\left[G^L(b), G^H(b)\right] \in [0,1] \times [0,1]$.

We start by defining the region in the plane where overlapping bidding supports are feasible, i.e. the region where (6) admits $g^L(b) \leq 0$ and $g^H(b) \leq 0$ (recall that $g^k(b) := -f_*^k(b)$). Since $a_1 > 0$, $a_2 > 0$ and $h < 0$, we see from (6) that $g^L(b) \leq 0$ holds always. For $g^H(b) \leq 0$, on the other hand, we need $a_4 > 0$ and $a_3 G^L(b) + a_4 G^H(b) \geq 0$, which outlines the region $\mathcal{D}$ where overlapping supports are feasible:

$$\mathcal{D} := \left\{ \left[G^L, G^H\right] \in [0,1] \times [0,1] : \frac{G^H}{G^L} \geq -\frac{a_3}{a_4} \right\}.$$

To derive explicit solutions for the bid distributions, we note that the matrix A has two real eigenvalues

$$\begin{aligned}
r_1 &= \frac{h}{2}\left(a_1 + a_4 - \sqrt{(a_1 - a_4)^2 + 4a_2 a_3}\right) < 0, \\
r_2 &= \frac{h}{2}\left(a_1 + a_4 + \sqrt{(a_1 - a_4)^2 + 4a_2 a_3}\right) < r_1 < 0.
\end{aligned}$$

To see that both r_1 and r_2 are real-valued, write

$$a_1 - a_4 = p_L(L)\, p_H(H)\left\{V_H(H) - V_L(L)\right\} + p_L(H)\, p_H(L)\left\{V_H(L) - V_L(H)\right\}$$

$$= p_L(L)\, p_H(H) \left\{ V_H(H) - V_L(H) \right\} + p_L(L)\, p_H(H) \left\{ V_L(H) - V_L(L) \right\}$$
$$+ p_L(H)\, p_H(L) \left\{ V_H(L) - V_L(L) \right\} - p_L(H)\, p_H(L) \left\{ V_L(H) - V_L(L) \right\}$$
$$\geq p_L(L)\, p_H(H) \left\{ V_H(H) - V_L(H) \right\} + p_L(H)\, p_H(L) \left\{ V_H(L) - V_L(L) \right\},$$

where the inequality follows from $V_L(H) \geq V_L(L)$ and log-supermodularity of p. As a result,

$$(a_1 - a_4)^2 \geq \left[p_L(L)\, p_H(H) \left\{ V_H(H) - V_L(H) \right\} + p_L(H)\, p_H(L) \left\{ V_H(L) - V_L(L) \right\} \right]^2.$$

Observe that $-a_2 a_3$ is simply the product of the two terms inside the bracket above:

$$-4a_2 a_3 = 4p_L(L)\, p_H(H)\, p_L(H)\, p_H(L) \left\{ V_H(H) - V_L(H) \right\} \left\{ V_H(L) - V_L(L) \right\}.$$

Since $(s+u)^2 \geq 4su$ for every $s, u \in \Re$, we have $(a_1 - a_4)^2 + 4a_2 a_3 \geq 0$.

The general solution to (6) is then

$$\begin{bmatrix} G^L(b) \\ G^H(b) \end{bmatrix} = c_1 \begin{bmatrix} k_1 \\ 1 \end{bmatrix} e^{r_1 b} + c_2 \begin{bmatrix} k_2 \\ 1 \end{bmatrix} e^{r_2 b}, \tag{7}$$

where c_1 and c_2 are free parameters and k_1 and k_2 are normalized eigenvector components

$$k_1 = \frac{a_1 - a_4 - \sqrt{(a_1 - a_4)^2 + 4a_2 a_3}}{2a_3} < 0,$$
$$k_2 = \frac{a_1 - a_4 + \sqrt{(a_1 - a_4)^2 + 4a_2 a_3}}{2a_3} < k_1 < 0.$$

We are now ready to analyze the three mutually exclusive cases in Proposition 1:

Case (i): $\phi_H(L) \geq \lambda \phi_L(L)$

In this case, $a_3 < a_4 \leq 0$ and therefore $\mathcal{D} = \emptyset$, meaning that the system (DE-2) does not have a feasible solution with $(f_*^L(b), f_*^H(b)) \gg 0$. Hence there should be no overlap between the two bid supports. The equilibrium (F_*^L, F_*^H) described in case (i) of Proposition 1 follows by solving the necessary indifference condition (DE-1) separately for F_*^L and F_*^H. First, (DE-1) for type $k = L$,

$$\frac{p_L(L)\, f_*^L(b)\, V_L(L)}{1 - p_L(L)\, F_*^L(b)} = 1,$$

with initial condition $F_*^L(0) = 0$ gives the formula for $F_*^L(b)$. Then setting the boundary condition $F_*^L(\overline{B}_L) = 1$ (recall that $\overline{B}_L = \sup \operatorname{supp}(F_*^L)$) gives the formula for $\overline{B}_L$. Similarly, (DE-1) for type

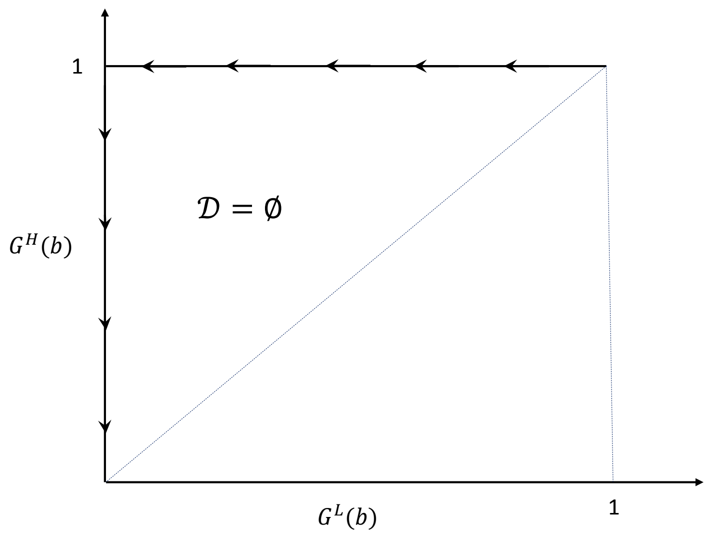


Figure 4: Case (i)

$k = H$ is

$$\frac{p_H(H) f_*^H(b) V_H(H)}{1 - p_L(L) - p_H(H) F_*^H(b)} = 1,$$

which with initial condition $F_*^H(\overline{B}_L) = 0$ gives the formula for $F_*^H(b)$. Figure 4 displays the equilibrium trajectory which is marked by arrows.

Case (ii): $\phi_L(L) \leq \phi_H(L) < \lambda \phi_L(L)$

In this case, we have $a_4 = -\phi_H(L) p_L(H) + \phi_L(L) p_H(H) > 0$ and

$$\left[G^L(b), G^H(b)\right] \in \mathcal{D} \quad \Leftrightarrow \quad \frac{G^H(b)}{G^L(b)} > -\frac{a_3}{a_4} = \frac{\phi_H(L) p_L(L) - \phi_L(L) p_H(L)}{\phi_L(L) p_H(H) - \phi_H(L) p_L(H)} > 1.$$

Since $G^H(0)/G^L(0) = 1 < -a_3/a_4$, bid zero cannot be in the support of F_*^H. It is easy to check that the only possibility is that there is a bid interval $[0, \underline{B}_H]$, where only type L is active, and $F_*^L(b)$ in that interval follows from the indifference condition (DE-1) for type L together with initial condition $F_*^L(0) = 0$. The formula for $\underline{B}_H$ is

$$\underline{B}_H = V_L(L) \log\left[\frac{\phi_H(L) - V_L(L) p_H(L)}{\phi_L(L) - V_L(L) p_H(L)}\right] < V_L(L) \log\left[p_L(H)\right]^{-1},$$

which follows from the condition

$$\frac{1}{G^L(\underline{B}_H)} = \frac{1}{1 - F_*^L(\underline{B}_H)} = -\frac{a_3}{a_4}$$

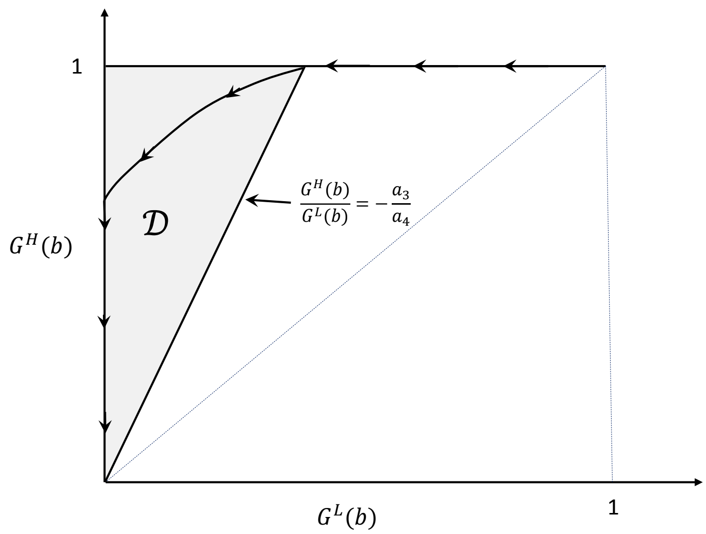


Figure 5: Case (ii)

that defines the point where the equilibrium path hits $\mathcal{D}$ where overlapping supports are feasible (see Figure 5, where $\mathcal{D}$ is the shaded area). It is easy to verify from (6) that we have a trajectory $\left[G^L(b), G^H(b)\right]$ with $g^L(b) < g^H(b) < 0$ for $b \in \left[\underline{B}_H, \overline{B}_L\right]$, where $\overline{B}_L$ is pinned down by $G^L\left(\overline{B}_L\right) = 1$. To get an explicit formula for the bidding distributions, we plug initial conditions $G^L\left(\underline{B}_H\right) = -\frac{a_4}{a_3}$ and $G^H\left(\underline{B}_H\right) = 0$ into (7), which gives:

$$\begin{aligned}
\widetilde{F}^L(b) &= 1 - G^L(b) = 1 - \widetilde{c}_1 \exp\left[r_1\left(b - \underline{B}_H\right)\right] - \widetilde{c}_2 \exp\left[r_2\left(b - \underline{B}_H\right)\right], \\
\widetilde{F}^H(b) &= 1 - G^H(b) = 1 - \widetilde{d}_1 \exp\left[r_1\left(b - \underline{B}_H\right)\right] - \widetilde{d}_2 \exp\left[r_2\left(b - \underline{B}_H\right)\right],
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{c}_1 &= \frac{\left(a_1 + a_4 + \sqrt{(a_1 - a_4)^2 + 4a_2a_3}\right)\left(a_1 - a_4 - \sqrt{(a_1 - a_4)^2 + 4a_2a_3}\right)}{4a_3\sqrt{(a_1 - a_4)^2 + 4a_2a_3}} < 0, \\
\widetilde{c}_2 &= -\frac{\left(a_1 + a_4 - \sqrt{(a_1 - a_4)^2 + 4a_2a_3}\right)\left(a_1 - a_4 + \sqrt{(a_1 - a_4)^2 + 4a_2a_3}\right)}{4a_3\sqrt{(a_1 - a_4)^2 + 4a_2a_3}} > 0, \\
\widetilde{d}_1 &= \frac{a_1 + a_4 + \sqrt{(a_1 - a_4)^2 + 4a_2a_3}}{2\sqrt{(a_1 - a_4)^2 + 4a_2a_3}} > 0, \\
\widetilde{d}_2 &= -\frac{a_1 + a_4 - \sqrt{(a_1 - a_4)^2 + 4a_2a_3}}{2\sqrt{(a_1 - a_4)^2 + 4a_2a_3}} < 0,
\end{aligned}$$

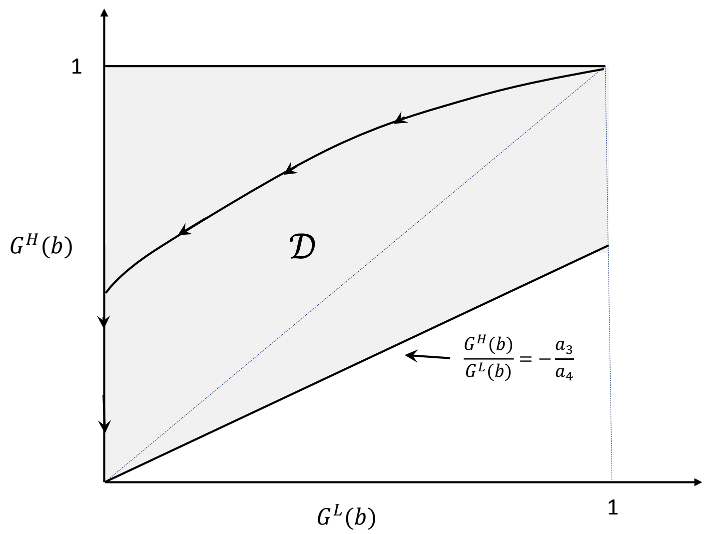


Figure 6: Case (iii)

and $\overline{B}_L$ is obtained from boundary condition $\widetilde{F}^L\left(\overline{B}_L\right) = 1$:

$$\overline{B}_L = \underline{B}_H + \left(-\frac{1}{h}\right) \cdot \frac{\log\left(\frac{a_4\left(a_1 - a_4 + \sqrt{(a_1-a_4)^2 + 4a_2a_3}\right) - 2a_2a_3}{a_4\left(a_1 - a_4 - \sqrt{(a_1-a_4)^2 + 4a_2a_3}\right) - 2a_2a_3}\right)}{\sqrt{(a_1-a_4)^2 + 4a_2a_3}}.$$

The remaining part of F_*^H for $b > \overline{B}_L$ follows directly from indifference condition (DE-1) for type H with initial condition $F_*^H\left(\overline{B}_L\right) = \widetilde{F}^H\left(\overline{B}_L\right)$.

Case (iii): $\phi_H(L) < \phi_L(L)$

In the last case, all points $\left[G^L(b), G^H(b)\right]$ satisfying $0 \leq G^L(b) \leq G^H(b)$ are in the set $\mathcal{D}$, including the initial point $\left[G^L(0), G^H(0)\right] = [1,1]$. The system (DE-2) then pins down a unique trajectory from $\left[G^L(0), G^H(0)\right] = [1,1]$ to $\left[G^L\left(\overline{B}_L\right), G^H\left(\overline{B}_L\right)\right]$, where $G^H\left(\overline{B}_L\right) > 0$ and where $\overline{B}_L$ defined by $G^L\left(\overline{B}_L\right) = 0$ (see Figure 6). Using initial conditions $G^L(0) = G^H(0) = 1$, we get the formulas for the bidding distributions from (7):

$$\begin{aligned} \widehat{F}^L(b) &= 1 - \widehat{c}_1 e^{r_1 b} - \widehat{c}_2 e^{r_2 b}, \\ \widehat{F}^H(b) &= 1 - \widehat{d}_1 e^{r_1 b} - \widehat{d}_2 e^{r_2 b}, \end{aligned}$$

where

$$\widehat{c}_1 = \frac{\left(a_1 - a_4 - 2a_3 + \sqrt{(a_1 - a_4)^2 + 4a_2a_3}\right)\left(a_1 - a_4 - \sqrt{(a_1 - a_4)^2 + 4a_2a_3}\right)}{4a_3\sqrt{(a_1 - a_4)^2 + 4a_2a_3}} < 0,$$

$$\widehat{c}_2 = \frac{\left(-a_1 + a_4 + 2a_3 + \sqrt{(a_1 - a_4)^2 + 4a_2a_3}\right)\left(a_1 - a_4 + \sqrt{(a_1 - a_4)^2 + 4a_2a_3}\right)}{4a_3\sqrt{(a_1 - a_4)^2 + 4a_2a_3}} > 0,$$

$$\widehat{d}_1 = \frac{a_1 - a_4 - 2a_3 + \sqrt{(a_1 - a_4)^2 + 4a_2a_3}}{2\sqrt{(a_1 - a_4)^2 + 4a_2a_3}} > 0,$$

$$\widehat{d}_2 = \frac{-a_1 + a_4 + 2a_3 + \sqrt{(a_1 - a_4)^2 + 4a_2a_3}}{2\sqrt{(a_1 - a_4)^2 + 4a_2a_3}} < 0,$$

and the highest bid $\overline{B}_L$ in the low-type bid support is determined by $\widehat{F}^L\left(\overline{B}_L\right) = 1$ as before:

$$\overline{B}_L = \left(-\frac{1}{h}\right) \cdot \frac{\log\left(\frac{\left(a_1 - a_4 - 2a_3 - \sqrt{(a_1 - a_4)^2 + 4a_2a_3}\right)\left(a_1 - a_4 + \sqrt{(a_1 - a_4)^2 + 4a_2a_3}\right)}{\left(a_1 - a_4 - 2a_3 + \sqrt{(a_1 - a_4)^2 + 4a_2a_3}\right)\left(a_1 - a_4 - \sqrt{(a_1 - a_4)^2 + 4a_2a_3}\right)}\right)}{\sqrt{(a_1 - a_4)^2 + 4a_2a_3}}.$$

Lastly, the part F_*^H for $b > \overline{B}_L$ can be obtained from indifference condition (DE-1) for type $k = H$ with initial condition $F_*^H\left(\overline{B}_L\right) = \widetilde{F}^H\left(\overline{B}_L\right)$. The proof is now complete.

References

Susan Athey. Single crossing properties and the existence of pure strategy equilibria in games of incomplete information. *Econometrica*, 69(4):861–89, July 2001. URL http://ideas.repec.org/a/ecm/emetrp/v69y2001i4p861-89.html.

Chang Koo Chi, Pauli Murto, and Juuso Välimäki. All-pay auctions with affiliated values. Technical report, Aalto University and HECER, June 2017.

Drew Fudenberg and Jean Tirole. A Theory of Exit in Duopoly. *Econometrica*, 54(4):943–960, July 1986. URL https://ideas.repec.org/a/ecm/emetrp/v54y1986i4p943-60.html.

Paul Klemperer and Jeremy Bulow. The Generalized War of Attrition. *American Economic Review*, 89(1):175–189, March 1999. URL https://ideas.repec.org/a/aea/aecrev/v89y1999i1p175-189.html.

Vijay Krishna and John Morgan. An Analysis of the War of Attrition and the All-Pay Auction. *Journal of Economic Theory*, 72(2):343–362, February 1997. URL http://ideas.repec.org/a/eee/jetheo/v72y1997i2p343-362.html.

John Maynard-Smith. The theory of games and the evolution of animal conflicts. *Journal of Theoretical Biology*, 47:209–221, 1974.

Paul Milgrom and Chris Shannon. Monotone comparative statics. *Econometrica*, 62(1):157–80, January 1994. URL http://ideas.repec.org/a/ecm/emetrp/v62y1994i1p157-80.html.

Paul R Milgrom and Robert J Weber. A theory of auctions and competitive bidding. *Econometrica*, 50(5):1089–1122, September 1982. URL http://ideas.repec.org/a/ecm/emetrp/v50y1982i5p1089-1122.html.